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論　文　目　録

報告番号	甲工 乙工　第 79 号 工修	氏　名	蒋　鈴　鴿
学位論文題目	Characteristic Curves of Nonlinear Resistive Circuits and their Stabilities		

論文の目次

備考
1 論文題目は, 用語が英語以外の外国語のときは日本語訳をつけて, 外国語, 日本語の順に列記すること。
2 参考論文は, 論文題目, 著者名, 公刊の方法及び時期を順に明記すること。
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参考論文

主論文

1. "Bifurcation Analysis of Nonlinear Resistive Circuits by Curve Tracing Method" Lingge JIANG and Akio Ushida, *IEICE Trans.*, vol.E-78A, no.9, pp.1225-1232, Sept. 1995.

副論文

1. "Analysis of Pulse Responses of Multi-Conductor Transmission lines by a Partitioning Technique" Yuichi TANJI, Lingge JIANG and Akio Ushida, *IEICE Trans.*, vol.E-77, no.12, pp.2017-2027, Dec. 1994.
2. "Stability of Characteristic Curves of nonlinear resistive circuits", Lingge JIANG and Akio Ushida, *Proc. 1995 Int. Symp. on Nonlinear Theor. and its Appl.*, vol.2, no.2, pp.1165-1170, Las Vegas, America, Dec. 1995.
3. "Analysis of Multi-Conductor Transmission lines Terminated by Nonlinear Subnetworks " Akio Ushida and Lingge JIANG *Proc. 1993 Joint Technical Conference on Circuits/Systems, Computers and Communications*, vol.1, no.2, pp.537-542, Nara, Japan, July, 1993.
4. "Stability of Nonlinear Resistive Circuits having Parasitic Elements" Lingge JIANG and Akio Ushida, *Tech. Report of IEICE* vol.NLP95-99, pp.7-14, Mar. 1996.
5. "Transient Analysis of Nonlinear Distributed Circuits by Relaxation Hybrid Harmonic Balance Method" Akio Ushida and Lingge JIANG, *Tech.Report of IEICE* vol.NLP92-11, pp.35-40, July, 1992.
6. "Stability of Nonlinear Reciprocal Resistive Circuits with Parasitic Elements" Lingge JIANG and Akio Ushida, *Proc. of IEICE 1996 National Conference*, vol.1, p.98, Tokyo, Mar. 1996.
7. "Analysis of Bifurcation Phenomena of Neural Network" Lingge JIANG and Akio Ushida, *Proc. of IEICE 1995 National Conference* vol.1, p.92, Fukuoka, Mar. 1995.
8. "Transient Analysis of Nonuniform interconnects by the Chebyshev Expansion Method" Lingge JIANG and Akio Ushida, *Proc. of IEICE 1994 National Conference*, vol.1, p.55, Yokohama, Mar. 1994.

備考

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電子回路の直流解析は回路設計における最も重要な問題の一つである。この場合、回路方程式は非線形連立代数方程式で記述され、この方程式の全ての解を求めることが必要である。このような問題は古くて新しいテーマであり、工学や数学の分野の多くの研究者によって盛んに研究が行われている。

ここで、次のよう非線形連立代数方程式を考えよう。

$$\left.\begin{array}{l} f_1(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ f_2(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \end{array}\right\}$$

この系は $n+1$ 個の変数をもつ n 個の方程式から構成されており、一般に、これを満足する解は n 次元空間における解曲線となる。ここで、x_{n+1}として、強制入力を選ぶならば、これを満足する解は駆動点特性曲線とか伝達特性曲線に対応する。また、x_{n+1}としてバイアス電圧を選び、$\{x_{n+1} : 0 \longrightarrow E_b\}$ と連続的に変化するならば、$x_{n+1} = E_b$において、直流解を得ることができる。この手法はニュートン法に比べて収束領域が広いために良く用いられる手法の一つである。

このような解曲線上には、各種の分岐点が存在する。特に、交差点 (交差点、ピッチフォーク点) では解曲線が複数個に分岐するが、このような解曲線を追跡 するためには分岐点において枝の方向を決定し、再び解曲線を追跡する必要がある。この様な解曲線追跡アルゴリズム開発を行った。一方、全ての抵抗素子には寄生素子が存在する。寄生素子値は抵抗素子値に比べて十分小さいと仮定できるが、大きさよりも、その存在が解曲線の安定性に関して非常に重要な役割を果たす。本研究では、さらに、分岐点において解曲線の安定性が変わることに注目し、解曲線の安定性に関する問題について議論したものである。

正帰還を内部にもつ電子回路は複数個の直流解が存在する可能性のあることが、Willson(California 大 Los Angeles 校) や西 (九大) によって提案されている。これらの回 路にはデイジタルの基本回路である多安定マルチやシュミットトリガ回路などが含まれている。これらの解はバイアス電圧が小さい範囲では1個であるが、バイアス電圧を増加して行くと、突然ある時点で複数個に分岐することが知られている。

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第二章では、篠原 (徳島大) による弧長法を用いた解曲線追跡法を提案している。そこでは、回路方程式は次のような

$$\left.\begin{array}{l} f_1(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ f_2(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ \left(\frac{dx_1}{ds}\right)^2 + \cdots + \left(\frac{dx_{n+1}}{ds}\right)^2 = 1 \end{array}\right\}$$

微分代数方程式で与えられる。この数値積分法として、ステイフな系に強い後退差分公式を導入して解曲線を追跡しようと云うのが本求解法である。この場合、係数行列は $(n+1)\times(n+1)$ となり、階数が n である限り連続的に追跡できるため、変曲点 (turning point) のような分岐点を含んだ曲線は連続的に追跡できる。また、この曲線は上の n 個の式を満足しているため得られた解曲線には数値積分による 蓄積誤差は存在しない。問題は交差点 (bifurcation point)、孤立点 (isolated point) 等が存在する場合である。このような点では解曲線追跡法の係数行列の階数が $(n-1)$ となり、特異行列となって追跡できなくなる。このような場合に、まず、正確な分岐点の位置を計算する必要ある。このような分岐点の計算法に対して、川上 (徳大)、山本 (九工大) などの方法があるが、ここでは、より単純な手法を提案している。次に、分岐点から複数個の枝方向を計算により求め、連続的に曲線を追跡しようと云うのが本解曲線追跡法である。

この解曲線追跡法は大域的な収束性をもつホモトピー法などにも適用される。一般に、ニューラルネットワークはニューロンの数を m とすると、2^m 個の平衡点が存在する。これらの解の個数はシグモイド関数に大きく依存する。本研究では、非線形度を未知変数に選び、解の分岐の様子を調べたものを例題として示してある。

第 3 章は、上のようにして得られた解曲線の安定性について議論したものである。抵抗回路には寄生キャパシタやインダクタが必ず存在する。それらは解の安定性に関して非常に重要な役割を果たす。従来から平衡点の安定性に関しては、変分方程式の固有値を計算することにより判定する方法が提案されているが、Willson (California 大、Los Angeles 校)、Green(Stony Brook 大) らは係数行列の行列式の値から安定性を判定する方法を提案している。

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本論文では、二つの安定判別手法を提案している。その一つは回路網トポジーに基づくものである。すなわち、“回路におけるある節点での微分コンダクタンスの和が負であるならば、その平衡点は不安定である。”この証明はリアプノフ理論に基づいている　。この場合、寄生素子は十分に小さいが、インダクタンスとキャパシタンスの比は零から無限大の範囲をとると仮定している。したがって、寄生素子の選び方によっては安定な場合もあることに注意されたい。

もう一つの安定判別法は二章の解曲線追跡手法に基づいている。すなわち、駆動点変数 x_{n+1} に対して

$$\frac{dx_{n+1}}{ds} = \frac{D_n}{D_{n+1}}$$

で与えられる。ここで、D_nは n 個の変数に対するヤコビ行列に対応し、D_{n+1}は解曲線追跡法における $(n+1)\times(n+1)$ の係数行列式である。したがって、通常の点では係数行列は正則である。一方 D_nの行列式の値は変分方程式の固有値の積に等しいので、解曲線の追跡方向が変わる毎に安定性が変わることが分かる。

しかしながら、交差点では、D_{n+1}が特異行列となるので、追跡方向が等しくても安　定性が変わることが分かった。一方、ピッチフォークの分岐点ではある軸に関して2つの枝が対称になるため、対称な曲線の枝の安定性は変わらない。

さらに、駆動点特性曲線において、$di_{n+1}/dv_{n+1} < 0$ は負性抵抗を意味し、その点から $\frac{dx_{n+1}}{ds} < 0$ まで の領域において、ホップ分岐が起こることが分かる。

これらの応用例として、2個のトンネルダイオードを持つ回路について、本手法による不安定領域をルンゲクッタ法による数値積分法を用いて実証した。また、フリップフロップ回路について、回路素子と分岐の様子、さらに安定性について調べた。その結果、回路が対称である時に限り、ピッチフォーク分岐の起こることが分かった。一方、対称な構造をもつホップフィールド回路では殆んどの分岐がピッチフォーク分岐であることが分かった。

第四章は、本研究の総括である、この解曲線追跡法と安定判別法は電子回路の直流解析に有効であることを示した。

Characteristic Curves of Nonlinear Resistive Circuits and their Stabilities

September 1996

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Characteristic Curves of Nonlinear Resistive Circuits and their Stabilities

Dissertation Submitted
in Candidacy for
the Ph. D. Degree

by

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September 1996

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Chapter 1

General Introduction

1.1 Background

Many problems arising in applications of physics, engineering or biology can be described by the mathematical models in the form of system of nonlinear ordinary differential equations which depend on real parameters.

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{\Lambda}) \tag{1.1}$$

where $\mathbf{x} \in R^n, \mathbf{\Lambda} \in R^m, \mathbf{f} : R^n \times R^m \to R^n$

$\mathbf{x}$ is a state variable vector, t is time, and $\mathbf{\Lambda}$ is generally a vector of system parameters. However, only a scalar parameter λ (m=1) will be considered throughout this study. The system is termed as autonomous, since the time variable does not occur in the right-hand side explicitly.

The stationary solutions of the model result from a set of nonlinear (algebraic) equations dependent on a chosen physical parameter.

$$\mathbf{f}(\mathbf{x}, \lambda) = \mathbf{0} \tag{1.2}$$

where $\mathbf{x} \in R^n, \lambda \in R, \mathbf{f} : R^n \times R \to R^n$. The sequence of solutions $(\mathbf{x}, \lambda)$ will consist of one or more solution curves in the $(n+1)$−dimensional Euclidean space.

For a given value of the parameter, the equations become as follows:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{1.3}$$

where $\mathbf{x} \in R^n, \mathbf{f} : R^n \to R^n$.

Solving above equations is very important in various practical engineering problems, because they are corresponding to equilibrium, DC solutions, etc. in practically.

There have been proposed a lot of algorithms studied on the topics.[1]-[8]. It is well known that Newton's method is usually used for solving (1.3)[8], however, there is a local convergence problem, i.e. the method is only convergence on such points where the initial guesses are chosen sufficiently close to the true solutions, while seeking such good initial guess is very difficult in practical computations. In order to overcome the difficulty, expand the convergence region, various homotopy methods have been introduced [2][7]. They possess a global convergence property and can be described as below.

First introduce a parameter $t \in [0, 1]$ and then construct the equation:

$$\mathbf{h}(\mathbf{x}, t) = \mathbf{0} \tag{1.4}$$

where $\mathbf{h}$ is defined by

$$\mathbf{h}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}) - (1-t)\mathbf{f}(\mathbf{x}^0) \tag{1.5}$$

The mapping $\mathbf{h}$ is termed as *homotopy* of (1.3). It is obvious that at $t = 0, \mathbf{h}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0) = \mathbf{0}$ give an obvious solution $\mathbf{x}^0$, and at $t = 1, \mathbf{h}(\mathbf{x}, 1) = \mathbf{f}(\mathbf{x}) = \mathbf{0}$, give the solution $\mathbf{x}^*$ we seek.

Depending on the different constructing method to $\mathbf{h}$, we can obtain many types of homotopies other than (1.5) which have the same property as above. Consider (1.5), since there is one more unknown than the number of equations, there are in general a continuum of points satisfied (1.5) so-called *homotopy path.* By tracing the path starting from $(\mathbf{x}^0, 0)$, we can obtain the solution $\mathbf{x}^*$ at $t = 1$.

We compare with (1.2) and (1.4) and find that the solutions of them are all present one or more curves in $n+1$-dimensional Euclidean space, which can be solved by continuation method. However, it is requirement to make distinction between them. For the former, there is a naturally occurring parameter, and hence all the values of the parameter are significant, the solution curves of themselves are important, while for the latter, the parameter has been artificially introduced, we are interested only in the terminal point $(\mathbf{x}^*, 1)$, other than the homotopy path. This distinction leads us adopting difference approaches to deal with them. Therefore the former is called continuation problem, and the latter is called homotopy problem in our study.

There are two fundamental types of algorithms for tracing a solution curves. One is based on piecewise-linearization such as the simplicial algorithm[5][7]. The second is by solving differential equations such as predictor-corrector algorithm[2][3][4][7][9][10].

Here we apply a curve tracing method based on the predictor-corrector algorithm to solve a continuation problem.

The *implicit function theorem* gives a condition under which the solution curve exists and is unique. That is the fundment of the continuation method.

Therom 1.1 *IMPLICIT FUNCTION THEOREM Let* $\mathbf{f} : R^{n+1} \to R^n$ *be continuously differentiable,* $(\bar{\mathbf{x}}, \bar{\lambda}) \in \mathbf{f}^{-1}$ *and* $\mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}}, \bar{\lambda})$ *be invertible. Then in a neighborhood of* $(\bar{\mathbf{x}}, \bar{\lambda})$ *all points* $(\mathbf{x}, \lambda)$ *that satisfy* $\mathbf{f}(\mathbf{x}, \lambda) = 0$ *are on a single continuously differentiable path through* $(\bar{\mathbf{x}}, \bar{\lambda})$

The structure of the solutions may change dramatically at certain critical points of a parameter, called bifurcation points[9][10].

To study the bifurcation points on solution curves is very important problem coming from follows view points: First, since the Jacobian of equations is degeneracy at bifurcation points, the curve tracing method will be fail at these points, in order to continue successfully the curve tracing process and obtain whole solution diagram, we have to study a behavior of branches in the neighborhood of the bifurcation point and evaluation of the directions of branches at bifurcation point. This is both very important to the continuation problem and the homotopy problem. Second, the bifurcation points are closely related to stability of practical system for continuation problem, which information will help us to design excellent system to be desired.

Obtaining whole solution curves diagram and investigating their stabilities are the common problem for various enginering designers. This filed is very intersting and still remains a lot of unsolved problems.

1.2 Purpose of this study

In this study we deal with the problems encountered in design of electronics system. We research the methods and algorithms for obtaining characteristic curve of nonlinear resistive circuits and investigating their stabilities.

Analysis of nonlinear resistive circuits is one of the central problems in electronic circuit system design, since an amount of electronic systems can be abstracted as some combination of nonlinear resistive circuits.

It is well known that the DC analysis of nonlinear resistive circuits is a fundamental and most important problem for electronic circuit CAD, where the solutions correspond to the operating points for a DC bias. when we solve a circuit equation described by n nonlinear equations in $(n+1)$ variables, the $(n+1)$th variable as some circuit parameter, then, since there is one more variables than the number of equations, the solutions will generally consist of one or more *solution curves* in the $(n+1)$-dimensional Euclidean space. If we select the $(n+1)$th variable as a DC bias or forced input, the characteristic curves: *driving point characteristic curve* or *transfer characteristic curve* for nonlinear resistive circuits can be obtained. The curves may have limit(turning) points, branching(crossing) points, and/or isolated points, which are called *bifurcation points.* Knowing the locations of these points is very important to understand the properties of circuit behavior, because the number of the solutions and their stabilities may be changed at the bifurcation points. But we cannot apply usual technique such as Newton-Raphson method to calculate them because at these points the rank of Jacobian matrix for a set of circuit equation is decreased by one or more, besides, when the solution curve is traced, it would be fail at these points by the same reason.

In chapter 2, we research computational methods for obtaining the bifurcation points and the branch directions at branching points of solution curves for the nonlinear resistive circuits. We first propose a simple modification technique such that the Newton-Raphson method can be also applied to the modified equations. Next, we show that our curve tracing algorithm can continuously trace the solution curves having the limit points and/or branching points. In this case, we can see whether the curve has passed through a bifurcation point or not by checking the sign of determinant of the Jacobian matrix. We also propose two different methods for calculating the directions of branches at branching point. Combining these algorithms, complicated solution curves

will be easily traced by the curve tracing method. A lot of numerical computation examples are given to exhibit the efficiency of the curve tracing algorithm. It is shown that the algorithm can be used successfully to tracing the solution curves of diode circuit, transistor circuit and Hopfield circuit which are all strong nonlinear circuits, at same time, the bifurcation points on the solution curves can also be located.

As previous description, the characteristic curves of nonlinear resistive circuits can obtained by solving a DC circuit equation which has n nonlinear equations in $(n+1)$ variables, however, its equilibrium point will be stable or unstable since every resistive element has a small parasitic component in practice[11][12]. It is an important information for designing various electronic circuits. A lot of efforts have been done about this filed[13]-[16], however, there are still a large amount of problems remained to be solved.

In chapter 3, we discuss the stability of the characteristic curves for nonlinear resistive circuits including parasitic elements. Although the DC solution is determined by analyzing the nonlinear resistive circuit, its equilibrium point will be the one of stable or unstable because every resistive element has a small parasitic component in practice. We consider here two parasitic elements: a *capacitor* between each resistor and ground, and an *inductor* in series with every resistor. Of course, the stability can be decided by solving the variational equation at each equilibrium point obtained by the DC analysis, however that is very time-consuming. Based on the Liapunov's direct method, we give some simple stability criterions for reciprocal resistive circuit and find that the stability may be changed at the boundary of the presence of negative differential resistance (NDR). For general circuits, we show here that the stability is *mainly* changed at the bifurcation points on the DC characteristic curves, so that the instability regions of the solution curve are easily found by the locations of bifurcation points on it. We summarize these into four definitions, three theorems and one corollary which are very useful to check the instability regions of the solution curves. Some illustrative examples are given in order to understand the theorems proposed in the paper.

We give conclusion and remark in chapter 4.

References

1. L.V.Kolev, "Find all solutions of nonlinear resistive equation via interval analysis ", *Int. J. Circuit Theory Appl.*, vol.12, pp.173-178, 1984.

2. L.O.Chua and A.Ushida, "A switching-parameter algorithm for finding multiple solutions of nonlinear resistive circuits", *Int. J. of Circuit Theory Appl.*, vol.4, pp. 215-239, 1976.

3. A.Ushida and L.O.Chua, "Tracing solution curve of nonlinear equations with sharp turning points", *Int. J. of Circuit Theory Appl.*, vol.12, pp.1-21, 1984.

4. C. He and A.Ushida, "A modified predictor-corrector tracing curve algorithm for solving nonlinear resistive circuits", *IEICE Trans.*, vol.E74, pp. 1455-1462, June 1991.

5. K. Yamamura and K. Horiuchi, "A globally and quadratically convergent algorithm for solving nonlinear resistive networks", *IEEE Trans.* Computer-Aided Design, Vol.9, No. 5, pp.487-499, May 1990.

   circuits: Computation of bifurcation values of system parameters," *IEEE Trans. Circuits Syst.*, vol.CAS-31, pp.248-260, Mar. 1984.

6. Y.Shinohara, "A geometric method of numerical solution of nonlinear equation and error estimation by Urabe's proposition", *RIMS Kyoto University*, vol.5, pp.1-9,1969.

7. C.B.Garcia and W.I.Zangwill, *Pathways to Solutions, Fixed Points, and Equilibria*, Prentice-Hall Englewood Cliff, N.J., 1981.

8. L.O.Chua and P.M.Lin, *Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques*, Prentice-Hall Englewood Cliff, N.J., 1975.

9. M.Kubicek and M.Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures*, Springer-Verlag, 1983.

10. L.G.Jiang and A.Ushida, "Bifurcation analysis of nonlinear resistive circuits by curve tracing method", *IEICE Trans.*,vol.E78-A, pp.1225-1232, Sept. 1995.

11. L.O.Chua and G.R.Alexander "The effects of parasitic reactances on nonlinear networks", *IEEE Trans. on Circuit Theory*, vol.CT-18, pp.520-531, Sept. 1971.

12. S.S.Sastry and C.A.Desoer "Jump behavior of circuits and systems", *IEEE Trans. on Circuit Theory*,vol.CAS-28, pp.1109-1124, Dec. 1981.

13. L.O.Chua and D.N.Green, "A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks", *IEEE Trans. Circuits Syst.*,vol.CAS-23, pp.355-530, June 1976.

14. M.M.Green and A.N.Willson,Jr, "How to identify unstable dc operating points", *IEEE Trans. Circuits Syst.*,vol.CAS-39, pp.821-832, Oct. 1992.

15. M.M.Green and A.N.Willson,Jr, "(Almost) half of any circuit's operating point are unstable", *IEEE Trans. Circuits Syst.*vol.CAS-41, pp286-293, April 1994.

16. M.M.Green and A.N.Willson,Jr, "An algorithm for identifying unstable operating points using SPICE", *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, vol.14, pp.360-370, March 1995.

Chapter 2

Characteristic Curve of Nonlinear Resistive Circuits

2.1 Introduction

In this study, we research computational methods for obtaining the bifurcation points and the branch directions at branching points of solution curves for the nonlinear resistive circuits. There are many kinds of the bifurcation points such as limit point, branch point and isolated point. At these points, the Jacobian matrix of circuit equation becomes singular so that we cannot directly apply the usual numerical techniques such as Newton-Raphson method. Therefore, we propose a simple modification technique such that the Newton-Raphson method can be also applied to the modified equations.

On the other hand, a curve tracing algorithm can continuously trace the solution curves having the limit points and/or branching points. In this case, we can see whether the curve has passed through a bifurcation point or not by checking the sign of determinant of the Jacobian matrix. We also propose two different methods for calculating the directions of branches at branching point. Combining these algorithms, complicated solution curves will be easily traced by the curve tracing method. We show some examples of various circuits to prove the efficiency of proposed algorithms.

2.2 To be solved problems

The DC analysis of nonlinear resistive networks is a fundamental and most important problem for electronic circuit design, where the solutions correspond to the operating points for a DC bias. There have been published many papers about algorithms for calculating the multiple solutions of nonlinear equations, and some of them can find all of the solutions [1-3]. Now, consider a circuit equation described by n nonlinear equations in $(n+1)$ variables

$$\left.\begin{array}{l} f_1(x_1, x_2, \cdots, x_n, x_{n+1}) = 0 \\ f_2(x_1, x_2, \cdots, x_n, x_{n+1}) = 0 \\ \\ f_n(x_1, x_2, \cdots, x_n, x_{n+1}) = 0 \end{array}\right\} \tag{2.1}$$

where x_{n+1} is sometimes chosen as a DC bias or forced input.

Since there is one more variables than the number of equations, the solution $\mathbf{x} = (x_1, x_2, \cdots, x_n, x_{n+1})^T$ satisfying (2.1) will generally consist of one or more *solution curves* in the $(n+1)$-dimensional Euclidean space. If we select x_{n+1} as a DC bias or forced input, then the characteristic curves: driving point curve or transfer curve for nonlinear resistive circuits can be obtained. The curves may have limit(turning) points, branching(crossing) points, and/or isolated points, which are called *bifurcation points*[4]. Knowing the locations of these points is very important to understand the properties of circuit behavior, because the number of the solutions and their stabilities [7] may be changed at the bifurcation points. But we cannot apply usual technique such as Newton-Raphson method to calculate them because at these points the rank of Jacobian matrix of (2.1) is decreased by one or more. Therefore, once the solution curve is traced, it would be fail at these points by the same reason. It is requirement that developing a method locating the bifurcation point and modifying curve tracing

algorithm to enhance the performance of dealing with various bifurcation points.

There are two methods for locating the bifurcation points, termed as direct method and indirect method. The former is a numerical method, i.e. to obtain the bifurcation points by solving an augmentative set of equations, which contain the original equations and the additional equation and they are nonsingular at the bifurcation points. The latter is a geometrical method, i.e. to locate the bifurcation points on solution curves by a curve tracing method. In this study, we research the both methods. There are some papers about algorithms for obtaining the bifurcation points. Kawakami[7] has proposed a method to find the limit point type bifurcation in the nonlinear periodic systems, where the equations are composed of original one (2.1) and the determinant of Jacobian matrix for first n variables $\{x_1, x_2, \ldots, x_n\}$. Kubicek and Marek [4] have also proposed a technique for calculating a branching point, where the rank of Jacobian matrix of (2.1) becomes $n-1$. Hence, they introduced one more equation from the determinant of Jacobian matrix for the variables $\{x_1, x_2, \ldots, x_{n-1}, x_{n+1}\}$. In this case, they solved the equations by the Gauss-Newton method [10] in the minimization approach, because there is one more equation than the variables, the convergence ratio is smaller than the Newton-Raphson method. Yamamoto [8] has proposed an elegant method for calculating the exact bifurcation point by the Newton-Raphson method, where additional variables are introduced and the equations are composed of the original one and their derivatives. We propose here another method combining Kubicek and Marek [4] and Yamamoto [8], where the equations are composed of the original one and the determinant of the Jacobian matrix. Thus, the number of equations is smaller than that of Yamamoto [8], and our method will be simpler for some kind of problems.

Besides the numerical method, we are also interested in a geometrical method. We show that our curve tracing algorithm can continuously trace the limit points and branching points, except for the special conditions such as the discretized step is exactly on the branching point. Furthermore, the sign of determinant of the Jacobian matrix is changed at the branching points. For the limit point, we propose another simple sign test. Thus, we can see whether the curve has passed through the bifurcation point or not by checking the sign.

After finding the branching point, we next calculate the direction of each branch. Thus, all of the solution curves will be continuously traced even if they have branching points.

2.3 Analysis of bifurcation points

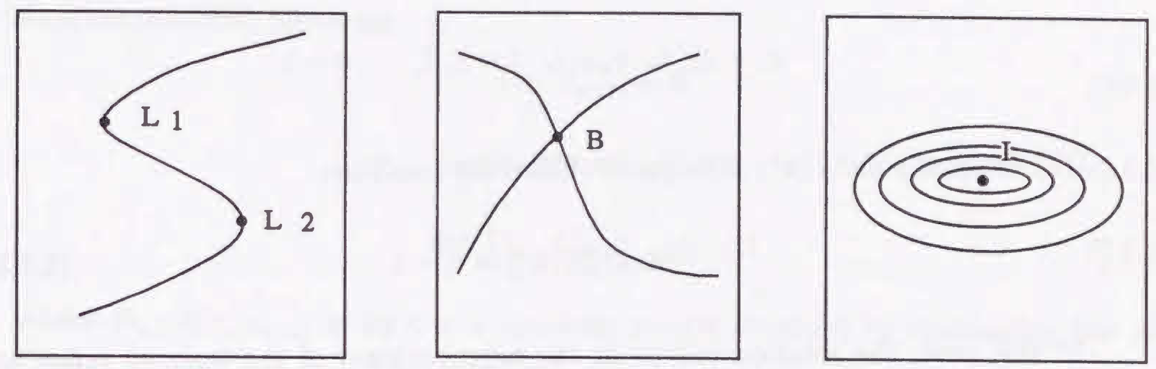
In order to derive the numerical algorithm, we first consider a classification of the bifurcation points and the description of their properties.. For simplicity, rewrite (2.1) as follows:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{2.2}$$

where $\mathbf{f} : R^{n+1} \to R^n$ is a C^2 nonlinear mapping in $\mathbf{x} \in R^{n+1}$. Let $D_n\mathbf{f}(\mathbf{x})$ be the Jacobian matrix for first n variables of $\mathbf{x}$. If it is nonsingular, the solution of (2.2) for a parameter x_{n+1} will be uniquely decided on the solution curve $\Gamma(\mathbf{x})$ by the *Implicity Function Theorem* given in Theorem 1.1. However, if the rank of Jacobian matrix

$$\mathrm{rank}(D_n\mathbf{f}(\mathbf{x})) \le n - 1 \tag{2.3}$$

at a solution point, then we will call it a *bifurcation point*. There are three types of bifurcation points called *limit point*, *branching point* and *isolated point* as shown in Fig. 2.1.



(i) Limit point (ii) Branching point (iii) Isolated point

Figure 2.1: Bifurcation points

Here, we consider the three types of bifurcations [4] as follows:

(i) *Limit point* $L(\mathbf{x}^*)$ satisfies the following conditions :

$$\mathrm{rank}(D_n\mathbf{f}(\mathbf{x})) = n - 1, \text{ but } \mathrm{rank}(D\mathbf{f}(\mathbf{x})) = n \tag{2.4.1}$$

where

$$D\mathbf{f}(\mathbf{x}) \equiv (D_n\mathbf{f}(\mathbf{x}), \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_{n+1}})$$

is the Jacobian matrix of (2.2) for $\mathbf{x} \in R^{n+1}$.

(ii) *Branching point* $B(\mathbf{x}^*)$ satisfies the following condition:

$$\mathrm{rank}(D\mathbf{f}(\mathbf{x})) = n - 1 \tag{2.4.2}$$

In this case, if the first $(n-1) \times (n-1)$ submatrix of the Jacobian matrix is nonsingular, the curves can be described by two variables x_n and x_{n+1} as follows:

$$x_i = \phi_i(x_n, x_{n+1}), \quad i = 1, 2, \ldots, n-1$$

(iii) *Isolated point* $I(\mathbf{x}^*)$ satisfies the following condition:

$$\mathrm{rank}(D\mathbf{f}(\mathbf{x})) = n - 1 \tag{2.4.3}$$

In this case, the solution curves in the neighborhood of the isolated point are described by

$$x_{n+1} = \phi(x_1, x_2, \ldots, x_n, x_{n+2})$$

for an additional parameter x_{n+2}, and it has an extremum at $I(\mathbf{x}^*)$.

Since the matrix $D_n\mathbf{f}(\mathbf{x}^*)$ is *singular* at these bifurcation points, some modifications must be done for applying the Newton-Raphson method to solve the singular points. First, for the *limit point*, it must satisfy two relations of (2.4.1). Thus, we have

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{2.5.1}$$

$$\det(D_n\mathbf{f}(\mathbf{x})) = 0 \tag{2.5.2}$$

If the Jacobian matrix of (2.5) is nonsingular at the limit point, we can calculate it by the Newton-Raphson method.[1]

For the *branching point* and *isolated point*, the rank of Jacobian matrix of (2.2) for $\mathbf{x} \in R^{n+1}$ is equal to $n-1$. Thus, we need to introduce an auxiliary variable x_{n+2}

[1] If the rank of Jacobian matrix of (2.5) will be still less than $n+1$, we need the further transformation for (2.5.2) as follows:

$$\det(D_n^k\mathbf{f}(\mathbf{x})) = 0$$

where k means kth order derivative by any variable $x_j, j \neq n+1$. We call the limit point as the *kth order limit point*. Note that the *inflection point* belongs to this type of limit point.

for finding the bifurcation point by the Newton-Raphson method. Now, consider the following modified relations:

$$\mathbf{f}(\mathbf{x}, x_{n+2}) = \mathbf{0} \tag{2.6.1}$$

$$\det(D_n\mathbf{f}(\mathbf{x}, x_{n+2})) = 0 \tag{2.6.2}$$

$$\det(D_{n+1}\mathbf{f}(\mathbf{x}, x_{n+2})) = 0 \tag{2.6.3}$$

where $D_{n+1}\mathbf{f}(\mathbf{x}, x_{n+2})$ is an $n \times n$ Jacobian matrix obtained by eliminating the jth column, $j \in [1, n]$ from the Jacobian matrix of (2.6.1). If the Jacobian matrix of the resulting equations (2.6) becomes nonsingular, then we can solve (2.6) by the Newton-Raphson method.

Of course, there are many techniques for introducing an auxiliary variable x_{n+2}, but they must satisfy some *necessary conditions* such that the rank of Jacobian matrix of (2.6.1) is increased by 1, and the relations (2.6) must have the same solution of (2.1) at $x_{n+2} = x^*_{n+2}$ [2].

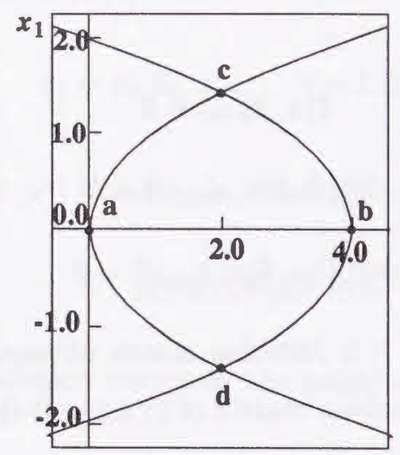
In order to obtain the higher order branching and isolated points, we need to replace the relations (2.6.2) and (2.6.3) by $\det(D_n^k\mathbf{f}(\ bfx))$ and $\det(D_{n+1}^k\mathbf{f}(\mathbf{x}))$, which have the same meaning as the case of the higher order limit point. Now, in order to understand well these proposed algorithms, we illustrate two simple examples.

Example2.1: Consider the following equation [4]

$$f_1(x_1, x_2) = ({x_1}^2 - x_2)({x_1}^2 + x_2 - 4) = 0$$

The solution curve is shown in Fig. 2.2, where the limit points are $a(0,0)$ and $b(0,4)$, the branching points are $c(\sqrt{2}, 2)$ and $d(-\sqrt{2}, 2)$. Differentiating $f_1(x_1, x_2)$ with respect

[2] Note that it is usually chosen $x^*_{n+2} = 0$ at bifurcation points. Ya mamoto [8] has introduced n-variables for the case of n equations, although one variable is fixed to 1. Thus, the number of augmented equations becomes $2n$.



a, b: Limit points c, d: Branching points

Figure 2.2: Example 2.1

to x_1 and x_2 respectively, we have

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= 2x_1(x_1^2+x_2-4)+2x_1(x_1^2-x_2)\\ \frac{\partial f_1}{\partial x_2} &= -(x_1^2+x_2-4)+(x_1^2-x_2)\end{aligned}$$

At the limit points a and b, we have

$$\frac{\partial f_1}{\partial x_1}=0,\quad \frac{\partial f_1}{\partial x_2}\neq 0$$

so that

$$\mathrm{rank}(Df_1(x_1,x_2))=\mathrm{rank}(\frac{\partial f_1}{\partial x_1},\frac{\partial f_1}{\partial x_2})=1$$

From (2.5.2), we have

$$f_1(x_1,x_2)=({x_1}^2-x_2)({x_1}^2+x_2-4)=0$$

$$\det(D_1f_1(x_1,x_2))=2x_1(x_1^2+x_2-4)+2x_1(x_1^2-x_2)=0$$

Thus, the Jacobian matrix at a,b is nonsingular and the limit points can be calculated by the Newton-Raphson method.

On the other hand, at the branching points c,d, we have

$$\frac{\partial f_1}{\partial x_1}=0,\quad \frac{\partial f_1}{\partial x_2}=0$$

In this case, the rank of Jacobian matrix $Df_1(x_1,x_2)$ is zero, so that we can not apply the Newton-Raphson method to them, directly.

Let us introduce an auxiliary variable x_3. Then, we have from (2.6)

$$\begin{aligned}f_1(x_1,x_2,x_3) &= (x_1^2-x_2)(x_1^2+x_2-4)+x_3=0\\ \det D_1f_1(x_1,x_2,x_3) &= 2x_1(x_1^2+x_2-4)+2x_1(x_1^2-x_2)=0\\ \det D_2f_1(x_1,x_2,x_3) &= -(x_1^2+x_2-4)+(x_1^2-x_2)=0\end{aligned}$$

The Jacobian matrix of above equations is given by

$$\begin{aligned}Df(x_1,x_2,x_3) &= \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3}\\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}\end{pmatrix}\\ &= \begin{pmatrix}4x_1^3-8x_1 & 4-2x_2 & 1\\ 12x_1^2-8 & 0 & 0\\ 0 & -2 & 0\end{pmatrix}\end{aligned}$$

It is obvious that the rank of Jacobian matrix becomes 3 at the branching points c and d, and then the Newton-Raphson method can be applied to solve these points.

Example 2.2: Consider an example of the isolated point given by

$$f_1(x_1,x_2)=x_1^4+x_2^4=0$$

which has the higher order singularity at the origin $(0,0)$. In this case, $D_1f_1(x_1,x_2)$

and $D_2 f_1(x_1, x_2)$ are given by

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 0$$

Now, introduce an auxiliary variable x_3 as follow:

$$f_1(x_1, x_2, x_3) = x_1^4 + x_2^4 - x_3 = 0$$

Then, the relations (6.2) and (6.3) for the higher order isolated point are given by

$$\det(D_1^3 f_1(x_1, x_2, x_3)) = \frac{\partial^3 f_1}{\partial x_1^3} = 24x_1 = 0$$

$$\det(D_2^3 f_1(x_1, x_2, x_3)) = \frac{\partial^3 f_1}{\partial x_2^3} = 24x_2 = 0$$

Thus, the Jacobian matrix for above three equations is given by

$$D\mathbf{f}(x_1, x_2, x_3) = \begin{pmatrix} 4x_1^3 & 4x_2^3 & -1 \\ 24 & 0 & 0 \\ 0 & 24 & 0 \end{pmatrix}$$

This is nonsingular at the isolated point, so that it can be solved by the Newton-Raphson method.

2.4 Curve tracing method

The bifurcation points can also be located when we trace the solution curves by a curve tracing method, which is termed as indirect method. The solution curves tracing methods can be rudely fallen into two categories. One is based on piecewise-linearization such as the simplicial algorithm[5], the other is by solving differential equations such as predictor-corrector algorithm[3][4][11]. Our curve tracing method belongs to the latter. The solution curves satisfying (2.1) can be solved by our curve tracing algorithm [3] starting from $\mathbf{x}_0$ on $(n+1)$-dimensional Euclidean space. They may have *limit points*

and/or *bifurcation points* on the curves. We first simply describe our curve tracing algorithm and then show that our curve tracing algorithm can check whether a curve has passed through the bifurcation point or not by a simple sign test.

2.4.1 Our curve tracing method

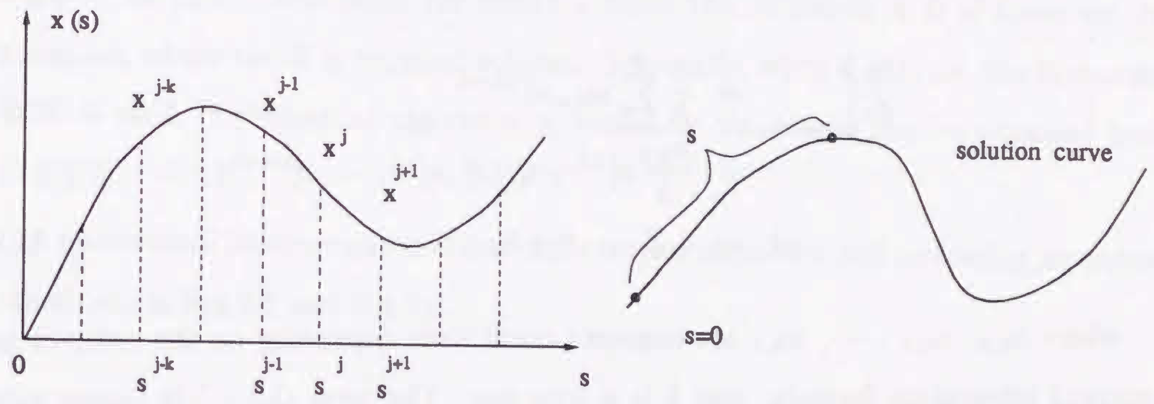


Figure 2.3: Arc-length description of solution curve

Let us describe the variable as $\mathbf{x} = \mathbf{x}(s)$ by a function of *arc-length s* from $\mathbf{x}_0$ shown as Fig.2.3. Then, we have from (2.2) the following set of algebraic-differential equations [3] :

$$\mathbf{f}(\mathbf{x}) = 0 \qquad (2.7.1)$$

$$\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \cdots + \left(\frac{dx_{n+1}}{ds}\right)^2 = 1 \qquad (2.7.2)$$

The set of algebraic-differential equations can be solved by one of the predictor-corrector algorithm. There are two different approaches which are dependent on the menner of dealing with the nonlinear differential equation (2.7.2), the one is based on solving the basic differential equation[4], and then substituting these into (2.7.2), the others embedded in our curve tracing method is using the backward difference formula to (2.7.2) as follows. Applying BDF (backward difference formula) algorithm [6], the kth order formula at $s = s^{j+1}$ is given by

$$\begin{aligned} \left.\frac{dx_i}{ds}\right|_{s=s^{j+1}} &= \frac{1}{h}\sum_{m=0}^{k}\alpha_{k,m}x_i^{j+1-m} \\ &= \frac{\alpha_{k,0}}{h}x_i^{j+1} + Q_k(x_i^j, x_i^{j-1}, \cdots, x_i^{j-k+1}) \\ i &= 1, 2, \ldots, n+1 \end{aligned} \tag{2.8}$$

where $\alpha_{k,0}$, $\alpha_{k,1}$,$\cdots$, $\alpha_{k,k}$ are constant coefficients depending on the order of numerical integration formula, and h is a step size. The term $Q_k(\cdots)$ is known value decided by the last k points on the solution curve. This is the BDF predictor algorithm, which offers a good guess obtained by extrapolating k previous solutions on the solution curves so that Newton-Raphson's method in the corrector step converges in several iterations.

Substituting (2.8) into (2.7.2), we have at the $(j+1)$th point the following algebraic equations :

$$\mathbf{f}(\mathbf{x}^{j+1}) = \mathbf{0} \tag{2.9.1}$$

$$\sum_{i=1}^{n+1}\left(\alpha_{k,0}x_i^{j+1} + hQ_k(x_i^j, x_i^{j-1}, \cdots, x_i^{j-k+1})\right)^2 - h^2 = 0 \tag{2.9.2}$$

$$i = 1, 2, \ldots, n+1$$

which is corrector equation and can be efficiently solved by the Newton-Raphson method if the Jacobian matrix is nonsingular.

We are interested in observing the geometrical means of the predictor-corrector algorithm (2.9), since there are all known value in (2.9.2) beside x_i^{j+1}, so (2.9.2) represents a $(n+1)$-dimensional sphere centered at $c = -\frac{hQ_k}{\alpha_{k,0}}$ with radius $r = \frac{h}{\alpha_{k,0}}$, as shown in Fig.2.4. Eq.(2.9.1) represents the solution curve, the intersects A, B of them are the solutions, where the B is required solution. Especially, when $k = 1$, i.e. the first-order BDF is used, the spherical algorithm proposed by Yamamura can be obtained from (2.9)[12].

A geometrical interpretations of our curve tracing algorithm and its tracing procedure are shown in Fig.2.5 and Fig.2.6.

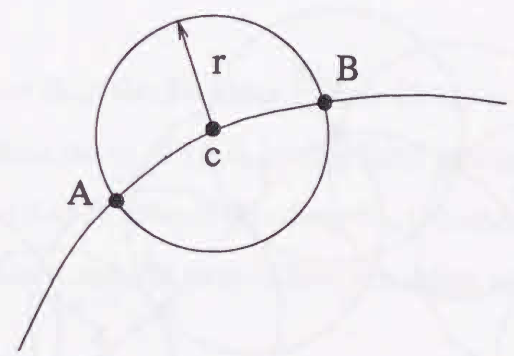


Figure 2.4: A sphere in $(n+1)$-dimension space

2.4.2 Properties at bifurcation points

We show that our curve tracing algorithm given by (2.9) can successfully trace the solution curve of (2.2) even when it has the limit point on $\Gamma(\mathbf{x})$. For simplicity, denote

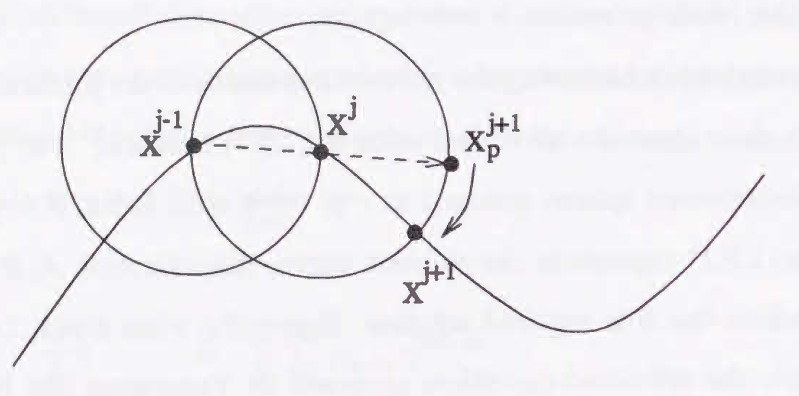


Figure 2.5: A geometrical interpretation of our curve tracing algorithm

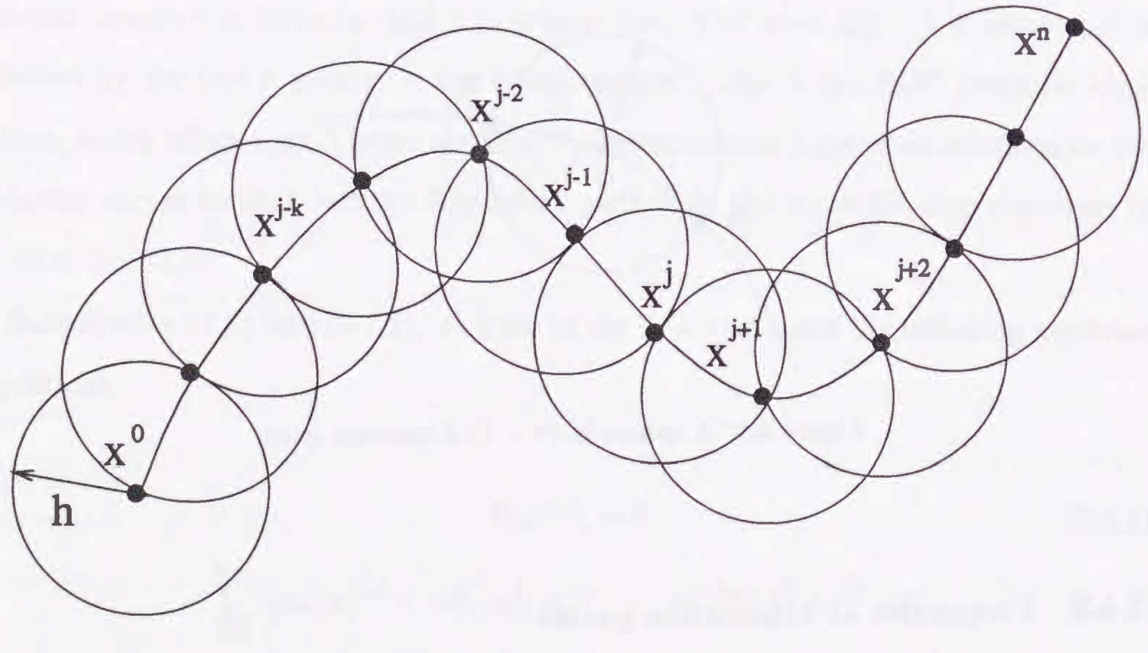


Figure 2.6: Curve tracing procedure

the algorithm (2.9) as follows:

$$\Gamma(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{f}(\mathbf{x}(s)) \\ N(\mathbf{x}(s)) \end{pmatrix} = \mathbf{0} \tag{2.10}$$

where $\mathbf{f}: R^{n+1} \to R^n$, $N: R^{n+1} \to R$, $\mathbf{x} = \mathbf{x}^{j+1} \in R^{n+1}$.

The Jacobian matrix of (2.10) is given by

$$D\Gamma(\mathbf{x}) = \begin{pmatrix} D_n\mathbf{f}(\mathbf{x}) & \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_{n+1}} \\ \mathbf{N}_n & N_{n+1,n+1} \end{pmatrix} \tag{2.11}$$

where

$$N_{n,i} = 2\alpha_{k,0}(\alpha_{k,0}x_i + hQ_k), \quad i = 1, 2, \ldots, n+1$$

Theorem 2.1 *If we choose the step size h sufficiently small, the Jacobian matrix of (2.9) is nonsingular at the limit points. Thus, our curve tracing algorithm (2.9) can continuously trace the solution curve of (2.2) even when it has the limit points on the solution curve* $\Gamma(\mathbf{x})$.

Proof: We want to prove that the Jacobian matrix (2.11) is nonsingular at the limit points. Consider a solution curve of (2.2) having limit points as shown in Fig.2.1(a). Since the curve is a continuous function of *arc-length s*, it is described by $x_i = x_i(s)$, $i = 1, 2, \ldots, n+1$. On the other hand, we have follows according to (2.2) and the arc-length method

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \\ \frac{dx_1}{ds} & \cdots & \frac{dx_n}{ds} & \frac{dx_{n+1}}{ds} \end{pmatrix} \begin{pmatrix} \frac{dx_1}{ds} \\ \cdot \\ \cdot \\ \frac{dx_n}{ds} \\ \frac{dx_{n+1}}{ds} \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} \tag{2.12}$$

Since dx_i/ds, $i = 1, 2, \ldots, n+1$ are bounded for s on the solution curve, the above equation must have a unique solution dx_i/ds. Therefore, the coefficient matrix of (2.12) must be nonsingular at the limit points. Now, substituting (2.8) into the last row of above coefficient matrix and multiplying to $2\alpha_{k,0}h$, then, we obtain the same matrix as the Jacobian matrix of (2.11). Thus, we have proved the theorem.

□

Note that although the determinant of the Jacobian matrix of (2.9) is always unequal to zero even at a limit point, which means that the Jacobian matrix of (2.9) never changes the sign when we trace the solution curves past through a limit point. However we can still distinguish whether the curve has passed through the limit point or not by only checking the sign of $\det D_n\mathbf{f}(\mathbf{x})$ from (2.5.2).

On the other hand, we have the following sign test at the branching point.

Theorem 2.2 *Let $\Gamma(\mathbf{x})$ be a smooth solution curve. Whenever the curve has passed through the branching point x^* in the region of $[s_k, s_{k+1}]$, the sign of $\det(D\Gamma(\mathbf{x}))$ is changed.* [3]

Proof. For simplicity, we set

$$d(s) = \det(D\Gamma(\mathbf{x}))$$

Now, applying Taylor expansion to $\det(D\Gamma(\mathbf{x}))$ at two points $s^*+\Delta s$ and $s^*-\Delta s$, we have

$$d(s^*+\Delta s) = d(s^*) + d'(s^*)\Delta s + \cdots \tag{2.13.1}$$

$$d(s^*-\Delta s) = d(s^*) - d'(s^*)\Delta s + \cdots \tag{2.13.2}$$

where $'$ means the derivative with respect to s. At the branching point $\mathbf{x}^*$, we have $\mathrm{rank}(D\mathbf{f}(\mathbf{x}^*)) = n-1$, and $d(s^*) = 0$.

Multiplying two equations of (2.13), we have

$$d(s^*+\Delta s)d(s^*-\Delta s) \simeq -[d'(s^*)]^2\Delta s^2$$

[3] Keller [9] has proved the same property by another way.

Thus, a sign of the right hand side is always changed to the negative whenever it has passed through the branching point.

□

Remark that since two solution curves are crossed at the branching point, our curve tracing algorithm may be failed to trace because the Jacobian matrix becomes singular at the point. Fortunately, in many numerical examples, our algorithm has safely passed through all the branching points because two curves are continuous except for the singular point x^*. However, when tangents of two curves are close to each other at the point, our method might be fail to trace.

2.5 Directions of branches at branching points

Now, consider a computational technique for deciding the directions of branches at a branching point. It is very important in order to obtain whole solution diagram because the curve tracing procedure continuous on only one branch bifurcating from the bifurcation point, and remaining branches are neglected, in practical numerical computations. Evaluation of the directions of branches at bifurcation points is considered as supporting menner for enhancing the efficiency of our curve tracing method. There are two methods. The first is an analytical method proposed by M.Kubicek and M.Marek [4], and the second is based on a geometrical technique.

2.5.1 Analytical method

We simply show the ideas and results of the analytical method in the reference [4]. Since the rank of Jacobian matrix of (2.2) is $n-1$ at a branching point, we can assume, without loss of the generality, that the first $(n-1)\times(n-1)$ submatrix is nonsingular at the branching point. Then, from the *implicit function theory*, the solution curve of

(2.1) can be described by

$$x_i = \phi_i(x_n, x_{n+1}), \quad i = 1, 2, \ldots, n-1$$

Applying Taylor expansion to $f_n(\mathbf{x})$ at the branching point $\mathbf{x}^*$ and substituting $x_i = \phi_i(x_n, x_{n+1}), i = 1, 2, \ldots, n-1$, we have

$$\begin{aligned} f_n(\mathbf{x}) &\simeq \frac{1}{2}A(x_n - x_n^*)^2 \\ &\quad + B(x_n - x_n^*)(x_{n+1} - x_{n+1}^*) \\ &\quad + \frac{1}{2}C(x_{n+1} - x_{n+1}^*)^2 = 0 \end{aligned} \tag{2.14}$$

where

$$A = \frac{\partial^2 f_n(x_n, x_{n+1})}{\partial x_n^2}, \quad B = \frac{\partial^2 f_n(x_n, x_{n+1})}{\partial x_n \partial x_{n+1}}$$

$$C = \frac{\partial^2 f_n(x_n, x_{n+1})}{\partial x_{n+1}^2}$$

Dividing (2.14) by $(x_n - x_n^*)^2$ or $(x_{n+1} - x_{n+1}^*)^2$, and taking the limits $x_n \to x_n^*$ and $x_{n+1} \to x_{n+1}^*$, we have

$$A\left(\frac{dx_n}{dx_{n+1}}\right)^2 + 2B\left(\frac{dx_n}{dx_{n+1}}\right) + C = 0 \tag{2.15.1}$$

$$C\left(\frac{dx_{n+1}}{dx_n}\right)^2 + 2B\left(\frac{dx_{n+1}}{dx_n}\right) + A = 0 \tag{2.15.2}$$

Solving Eq.(2.15), the directions of branches are evaluated as follows:

Case 1: When $A \neq 0$ and $B^2 - AC > 0$, we have from (2.15.1)

$$\left.\frac{dx_n}{dx_{n+1}}\right|_{x=x^*} = \frac{-B \pm \sqrt{B^2 - AC}}{A} \tag{2.16.1}$$

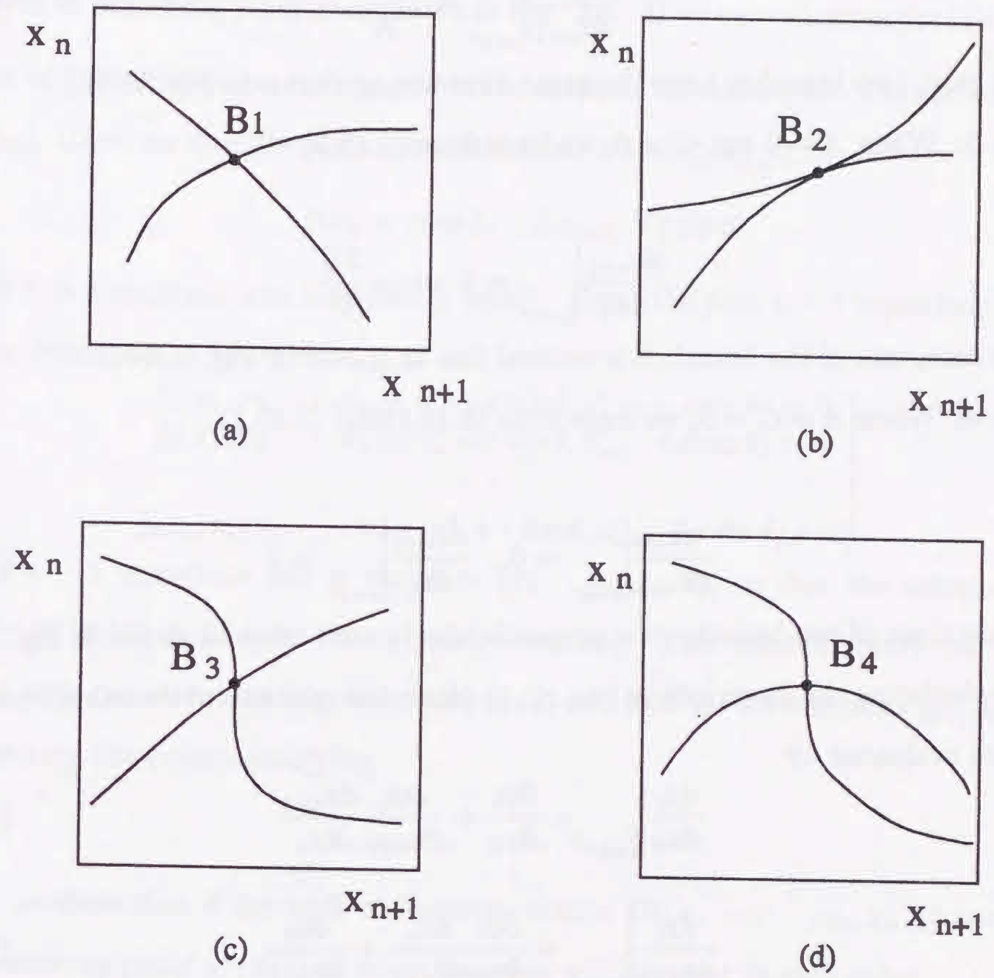


Figure 2.7: Directions of branches at branching points

Hence, two branches have different directions as shown in Fig. 2.7(a).

Case 2: When $A \neq 0$ and $B^2 - AC = 0$, we have from (2.15.1)

$$\left.\frac{dx_n}{dx_{n+1}}\right|_{x=x^*} = -\frac{B}{A} \tag{2.16.2}$$

In this case, two branches have the same direction as shown in Fig. 2.7(b).

Case 3: When $A = 0$ but $C \neq 0$, we have from (2.15.2)

$$\left.\frac{dx_{n+1}}{dx_n}\right|_{x=x^*} = 0 \text{ or } -\frac{2B}{C} \tag{2.16.3}$$

In this case, one of the branch is a vertical line as shown in Fig. 2.7(c).

Case 4: When $A = C = 0$, we have from (2.15.1)and (2.15.2)

$$\left.\frac{dx_n}{dx_{n+1}}\right|_{x=x^*} = 0, \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{x=x^*} = 0 \tag{2.16.4}$$

The directions of two branches are perpendicular in each other as shown in Fig. 2.7(d).

After calculating the directions in (x_n, x_{n+1})-plane, the relations of the other coordinate axes are evaluated by

$$\left.\frac{dx_i}{dx_n}\right|_{x=x^*} = \frac{\partial\phi_i}{\partial x_n} + \frac{\partial\phi_i}{\partial x_{n+1}}\frac{dx_{n+1}}{dx_n} \tag{2.17.1}$$

or

$$\left.\frac{dx_i}{dx_{n+1}}\right|_{x=x^*} = \frac{\partial\phi_i}{\partial x_n}\frac{dx_n}{dx_{n+1}} + \frac{\partial\phi_i}{\partial x_{n+1}} \tag{2.17.2}$$

Remark that although the ideas given above are elegant, we need a complicated and troublesome computations for getting A, B and C. Next, we show a very simple method based on a geometrical technique.

2.5.2 Geometrical method

We assume that, at the branching point, the first $(n-1) \times (n-1)$ submatrix of Jacobian matrix of (2.2) is nonsingular. Now, consider a small cylinder of radius r centered at branching point B as shown in Fig. 2.8. If we can calculate the intersection points of the solution curves on cylindrical surface, the directions of branches will be decided. Here, we describe small variation from $B = (x_n^*, x_{n+1}^*)$ by

$$\Delta x_n = r\cos\theta, \quad \Delta x_{n+1} = r\sin\theta \tag{2.18}$$

where r is a constant and sufficiently small. Then, the first $n-1$ equations of (2.1) can be described as follows:

$$\left.\begin{array}{r} f_1(x_1, x_2, \cdots, x_{n-1}, x_n^* + r\cos\theta, x_{n+1}^* + r\sin\theta) = 0 \\ f_2(x_1, x_2, \cdots, x_{n-1}, x_n^* + r\cos\theta, x_{n+1}^* + r\sin\theta) = 0 \\ \vdots \\ f_{n-1}(x_1, x_2, \cdots, x_{n-1}, x_n^* + r\cos\theta, x_{n+1}^* + r\sin\theta) = 0 \end{array}\right\} \tag{2.19}$$

It has $n-1$ equations and n variables $(x_1, \ldots, x_{n-1}, \theta)$, so that the solution curve satisfying (2.19) is laid on the cylindrical surface. This solution curve can be also traced with our curve tracing algorithm (2.7), and find the intersections (p_1, p_2, p_3, p_4) by checking the points satisfying

$$f_n(x_1, x_2, \cdots, x_n, x_{n+1}) = 0$$

Next, we show that if the rank of Jacobian matrix $D\mathbf{f}(x_1, x_2, \cdots, x_n, x_{n+1})$ is $n-1$ at the branching point $\mathbf{x}^*$, two or more branches will intersect in each other.

Applying Taylor expansion to (2.19), we have

$$\left.\begin{array}{r} h_{11}\Delta x_1 + \cdots + h_{1,n} r\cos\theta + h_{1,n+1} r\sin\theta = 0 \\ h_{21}\Delta x_1 + \cdots + h_{2,n} r\cos\theta + h_{2,n+1} r\sin\theta = 0 \\ \vdots \\ h_{n-1,1}\Delta x_1 + \cdots + h_{n-1,n} r\cos\theta + h_{n-1,n+1} r\sin\theta = 0 \end{array}\right\} \tag{2.20}$$

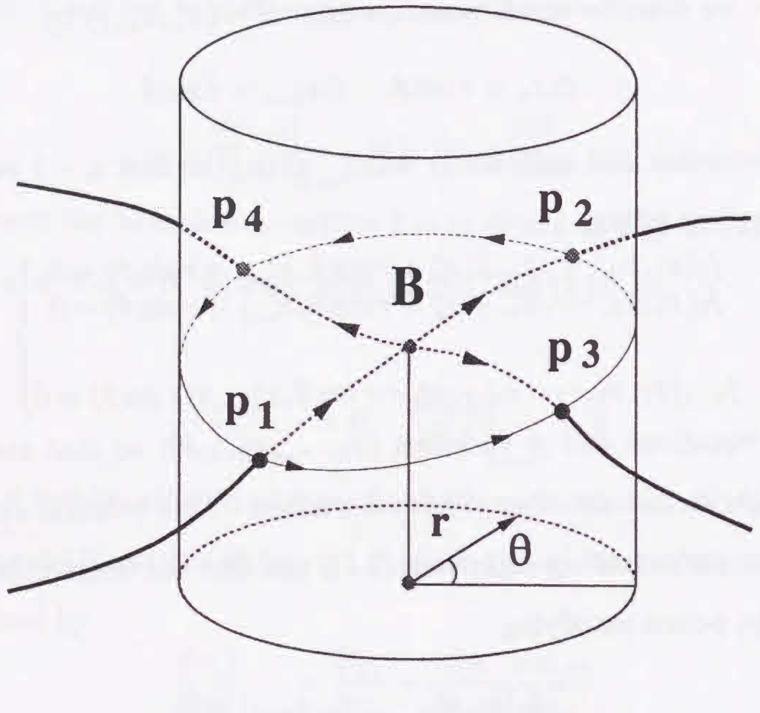


Figure 2.8: Geometrical method for deciding the directions of branches

We also have from $f_n(\mathbf{x})$

$$\begin{aligned} & f_n(x_1^* + \Delta x_1, x_2^* + \Delta x_2, \cdots, x_{n+1}^* + \Delta x_{n+1}) \\ & = \sum_{i=1}^{n-1} h_{n,i}\Delta x_i + h_{n,n} r\cos\theta + h_{n,n+1} r \sin\theta \\ & + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \frac{\partial^2 f_n}{\partial x_i \partial x_j}\Delta x_i \Delta x_j \end{aligned} \tag{2.21}$$

Substituting $\Delta x_1, \Delta x_2, \ldots, \Delta x_{n-1}$ from (2.20) into (2.21), we can easily prove that the sum of first 3 terms in the right side is zero. On the other hand, the solutions of (2.21) can be written as follows:

$$\begin{aligned} \Delta x_i &= (\varepsilon_{2i-1}\cos\theta + \varepsilon_{2i}\sin\theta) r \\ i &= 1, 2, \ldots n-1 \end{aligned} \tag{2.22}$$

Substituting (2.22) into (2.21), we have

$$\begin{aligned} & f_n(x_1, x_2, \ldots, x_{n+1}) \\ & = (C_{2n-1}\cos 2\theta + C_{2n}\sin 2\theta + C_0) r^2 = 0 \end{aligned} \tag{2.23}$$

where C_0, C_{2n-1}, C_{2n} are constant.

It is clear from (2.21) or (2.23) that f_n is the function of 2θ so that it has 0, 2 or 4 solutions on the cylindrical surface. [4]

[4] It is difficult to know the number of solutions satisfying (2.23) from the theoretical point of view. For example of the isolated point, the Hessian matrix given by ($h_{ij} = \frac{\partial^2 f_n}{\partial x_i \partial x_j}$) in (2.21) is positive or negative definite. Hence, the relation (2.23) will not have solution. On the other hand, for the branching point, two curves cross as shown in Fig. 2.7, so that they will intersect at 2 or 4 points on the cylindrical surface. Thus, (2.23) will have 2 or 4 solutions in this case.

Example 2.3: Consider the following equation

$$f(x_1, x_2) = (x_1^2 - x_2)(x_1^2 + x_2 - 4) = 0 \tag{2.24.1}$$

At a branching point $c(\sqrt{2}, 2)$ in Fig. 2.2, we have

$$x_1 = x_1^* + \Delta x_1 = x_1^* + r\cos\theta \tag{2.24.2}$$

$$x_2 = x_2^* + \Delta x_2 = x_2^* + r\sin\theta \tag{2.24.3}$$

Applying Taylor expansion to (2.24.1), and substituting (2.24.2) and (2.24.3) into it, we have

$$(4x_1^{*3} - 8x_1^*)r\cos\theta + (-2x_2^* + 4)r\sin\theta$$
$$+(6x_1^{*2} - 4)r^2\cos^2\theta - r^2\sin^2\theta = 0$$

Thus

$$8r^2\cos^2\theta - r^2\sin^2\theta = 0$$

Therefore, it has 4 solutions in $\theta = [0, 2\pi]$ as follows:

$$\tan\theta = 2\sqrt{2}, -2\sqrt{2}$$

2.6 The flowchart of tracing a characteristic curve

Till now, we have discussed the computational methods for obtaining the bifurcation points and the branch directions at branching points of solution curves for the nonlinear resistive circuits. First, the bifurcation points were classified as *limit point, branching point* and *isolated point* and the mathematical definitions were given. Based on these properties, we proposed two method for obtaining the bifurcation points, which are

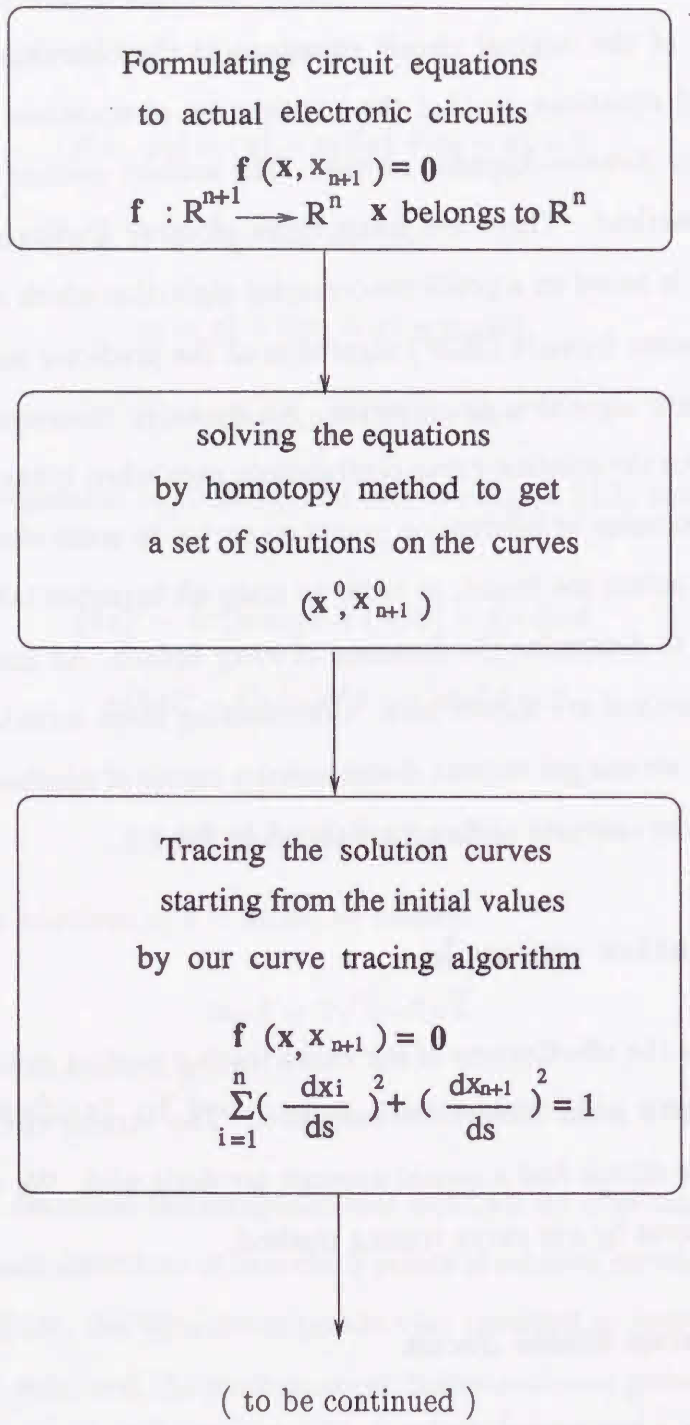
termed as direct method and indirect method. The idea of direct method is removing the singularity of the original circuit equations at the bifurcation points by adding some associated equations, so that the augment set of equations become regular and can be solved by Newton-Raphson method. The indirect method is dependent on the curve tracing method. There are many types of curve tracing method. Our curve tracing method is based on a predictor-corrector algorithm which applies the kth order backward difference formula (BDF) algorithm as the predictor and uses the Newton-Raphson iteration algorithm as corrector. As shown in theorems, our curve tracing method can trace the solution curve continuously even when it has limit points and we can decide the location of bifurcation points on curves by some simple sign tests. Once the bifurcation points are found, in order to trace all branches bifurcating from these points, we have to determine the direction of every branch. An analytical method and a geometrical method are shown here. Combinating these techniques into our curve tracing method, we can get various characteristics curves of nonlinear resistive circuits. We summarize the contents as flowchart shown in Fig.2.9.

2.7 Illustrative example

In order to verify the effectiveness of our curve tracing method embedded the modified techniques, we give some illustrative examples. The various circuits such as diode circuit, transistor circuit and a neural network are dealt with. We want to get various characteristic curves by our curve tracing method.

2.7.1 Two-tunnel diodes circuit

Now, we consider a two tunnel diodes circuit shown as below.



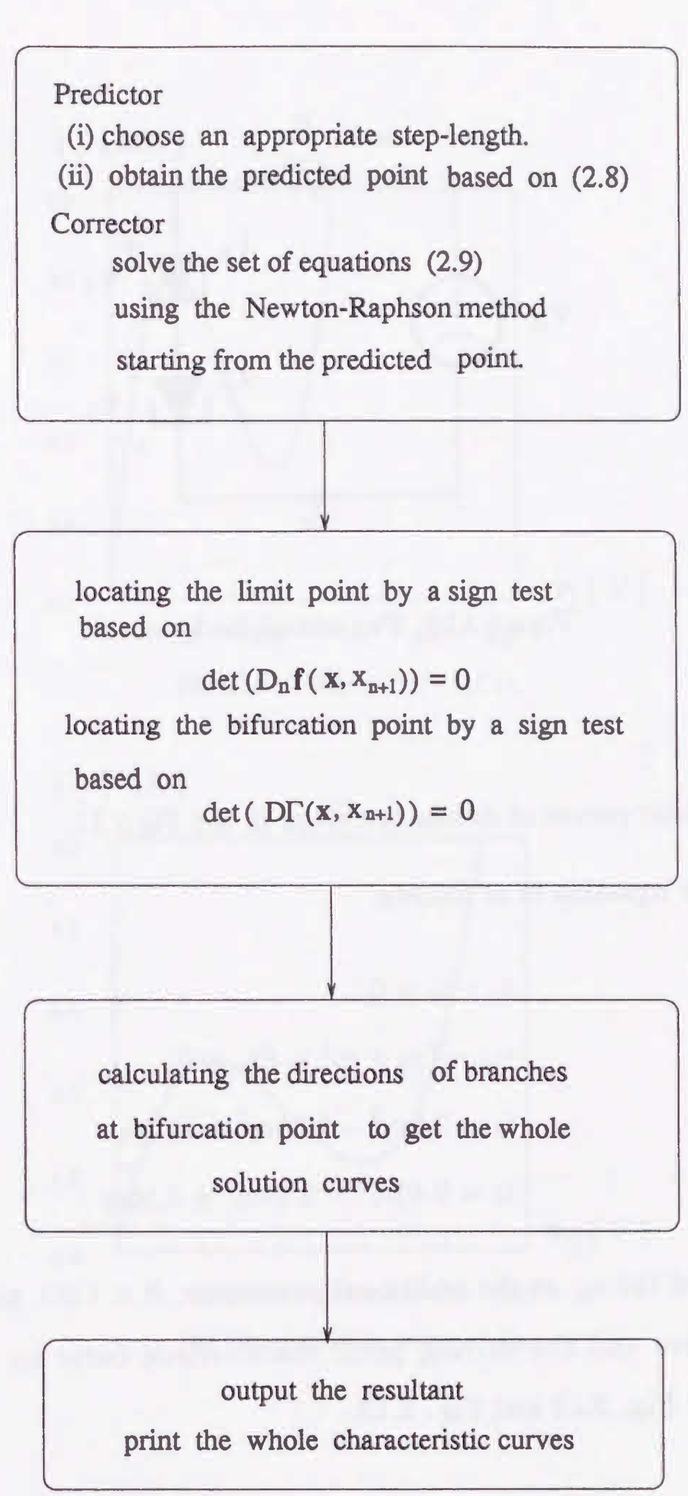


Figure 2.9: Flowchart of obtaining the characteristic curves

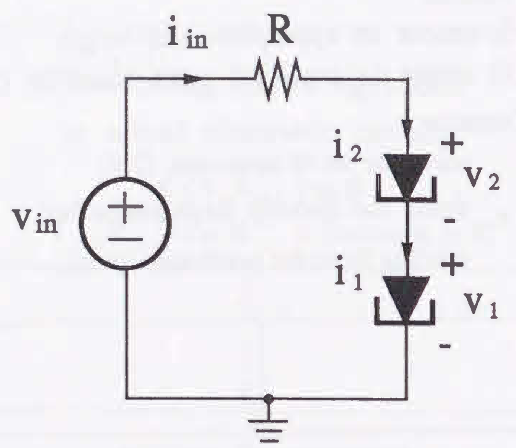


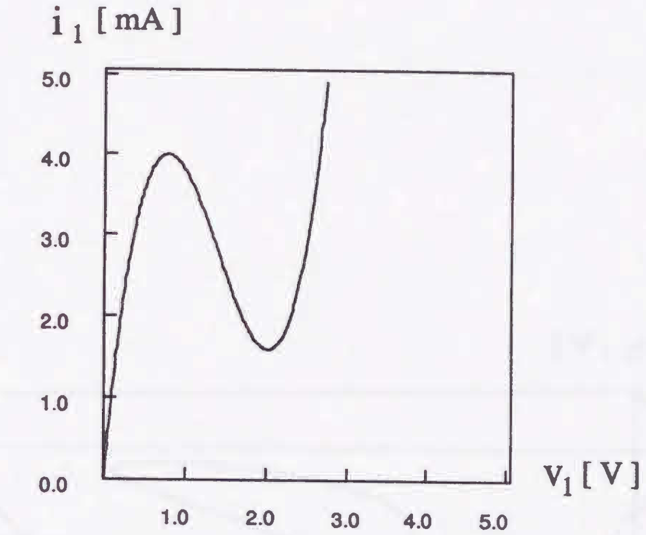
Figure 2.10: Two tunnel diodes circuit

The characteristic curves of diodes are given in the Fig.2.11.

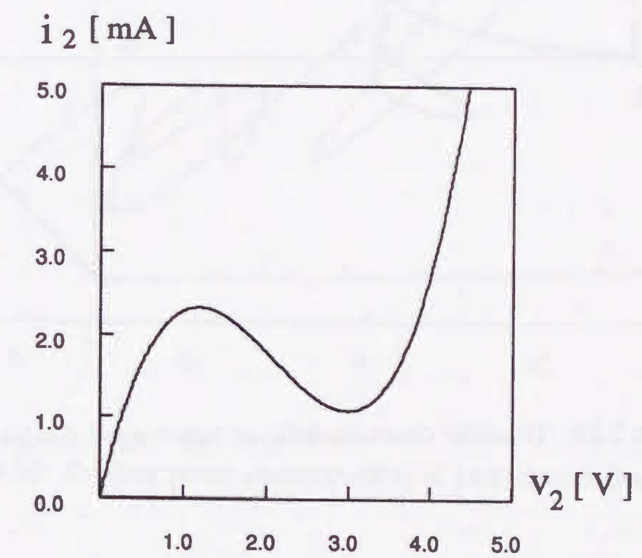
The DC circuit equation is as follows:

$$
\begin{aligned}
& i_1 - i_2 = 0 \\
& v_{in} - (v_1 + v_2) - Ri_1 = 0 \\
& i_1 = 2.5{v_1}^3 - 10.5{v_1}^2 + 11.8v_1 \\
& i_2 = 0.43{v_2}^3 - 2.69{v_2}^2 + 4.56v_2
\end{aligned}
\tag{2.25}
$$

Here we selected the v_{in} as the additional parameter, $R = 1.5\Omega$, and got the transfer characteristic curve and the driving point characteristic curve by our curve tracing method shown as Fig. 2.12 and Fig. 2.13.



(a) Characteristic curve of D_1



(b) Characteristic curve of D_2

Figure 2.11: Characteristic curves of diodes

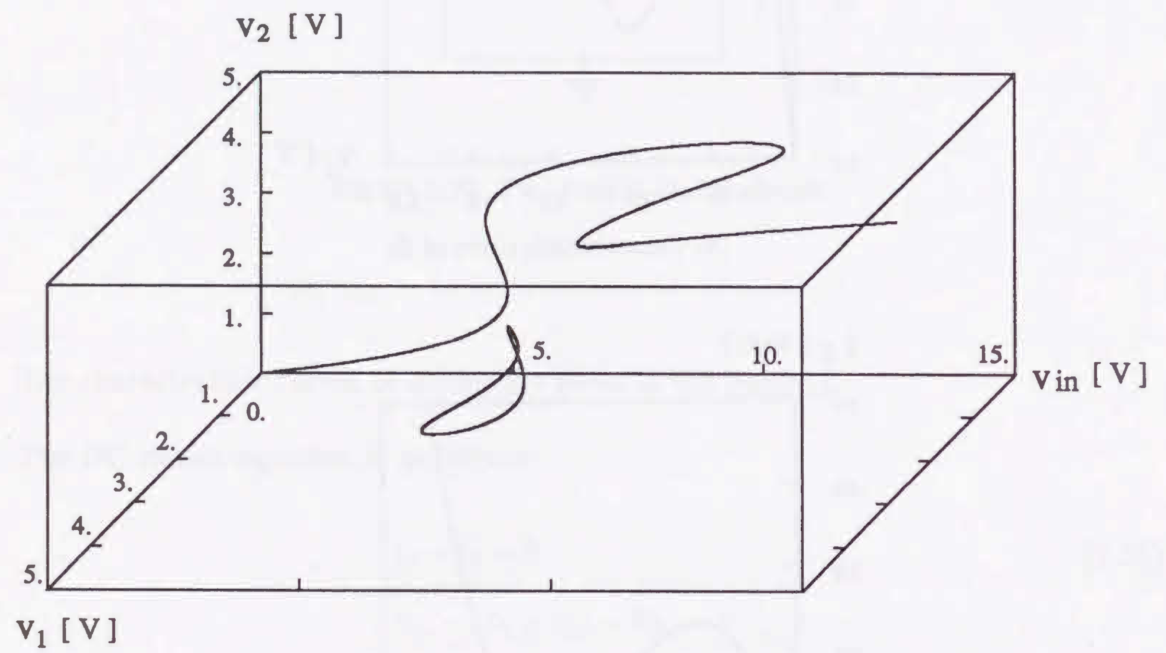


Figure 2.12: Transfer characteristic of two tunnel diodes circuit

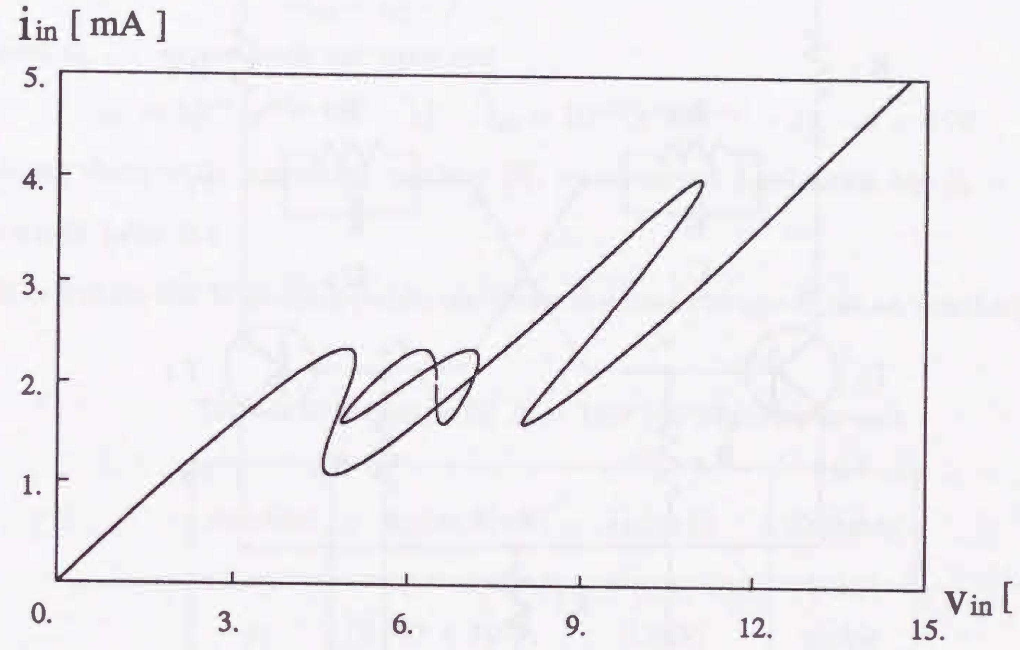


Figure 2.13: Driving point characteristic of two tunnel diodes circuit

2.7.2 Flip-Flop circuit

Now, consider a Flip-Flop circuit as shown in Fig. 2.14.

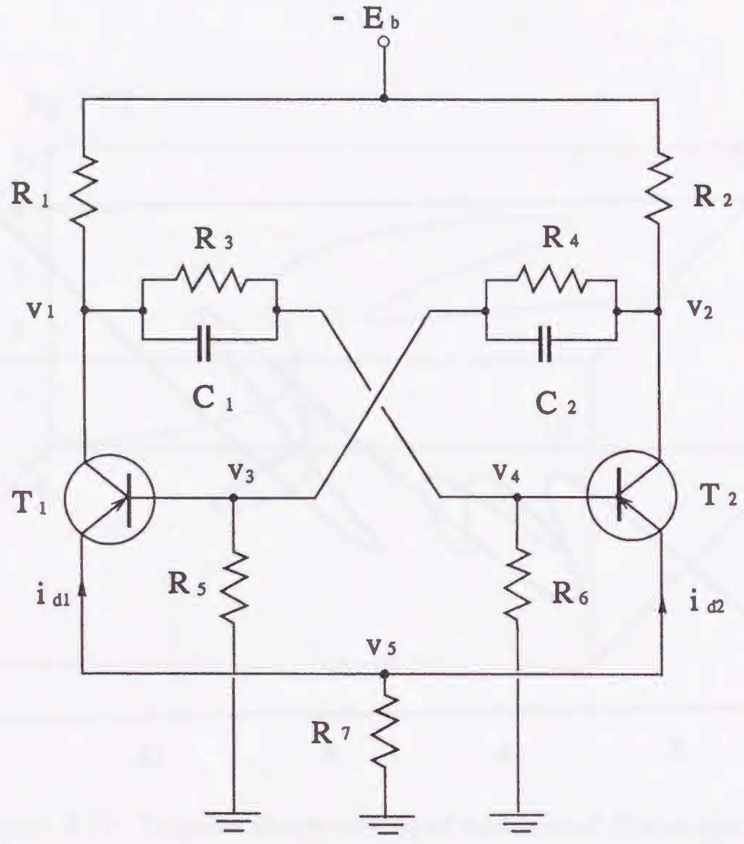


Figure 2.14: Flip-Flop circuit

where $R_1 = R_2 = 15K\Omega,\quad R_3 = R_4 = 140K\Omega,\quad R_5 = R_6 = 10K\Omega,$

$R_7 = 0.5K\Omega,\quad C_1 = C_2 = 100pF.$

Modeling two transistors by Ebers-Moll model, the circuit equations can be written as

follows:

$$\begin{pmatrix} G_1+G_3 & 0 & 0 & -G_3 & 0 \\ 0 & G_2+G_4 & -G_4 & 0 & 0 \\ 0 & -G_4 & G_4+G_5 & 0 & 0 \\ -G_3 & 0 & 0 & G_3+G_6 & 0 \\ 0 & 0 & 0 & 0 & G_7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} \alpha i_{d1} - G_1 E_b \\ \alpha i_{d2} - G_2 E_b \\ i_{d1} - \alpha i_{d1} \\ i_{d2} - \alpha i_{d2} \\ -i_{d1} - i_{d2} \end{pmatrix}$$

where $v_1, \ldots, v_5$ are node voltages and

$$i_{d1} = 10^{-9}(e^{40(v_5 - v_3)} - 1), \quad i_{d2} = 10^{-9}(e^{40(v_5 - v_4)} - 1), \quad \alpha = 0.98$$

Solving them with homotopy method [3], we obtained 3 solutions for $E_b = 12(V)$ as shown in table 2.1.

To calculate the branching point, we chose the bias voltage E_b as an auxiliary parameter, and found the bifurcation point B on solution curve at $E_b = 8.39(V)$ as shown in Fig. 2.15.

Table 2.1: Solutions for $E_b = 12(V)$ of Flip-Flop circuit

solution	$i_{d1}(mA)$	$i_{d2}(mA)$	*Stability*
p_1	$(0.222 \times 10^{-9},$	$0.343)$	stable
p_2	$(0.126,$	$0.126)$	unstable
p_3	$(0.343,$	$0.222 \times 10^{-9})$	stable

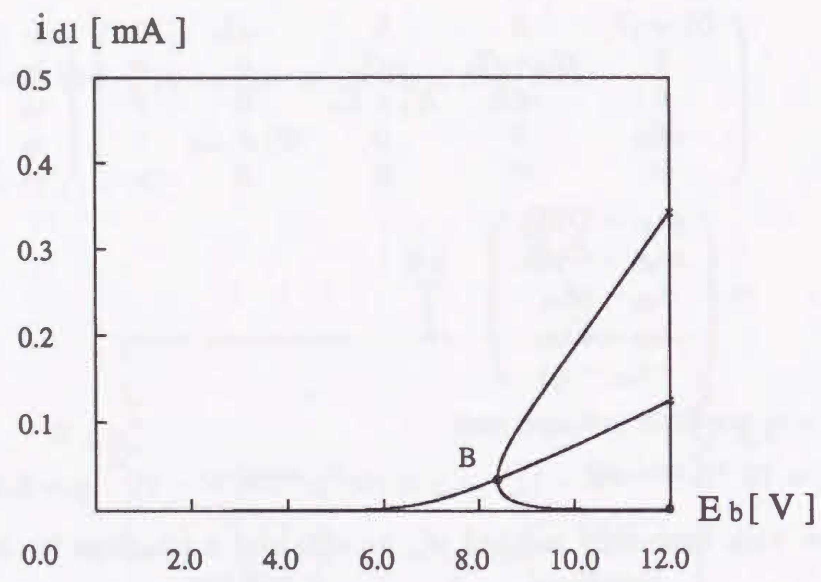


Figure 2.15: Bifurcation diagram of the Flip-Flop circuit

The bifurcation diagram of the Flip-Flop circuit shows that if we choose the bias voltage less than the E_b, the flip-flop circuit will never work well, since there is only one steady state. Thus, the example helps us to understand that the calculation of bifurcation point is very important to design the electronic circuits exactly.

2.7.3 Hopfield networks

A Hopfield neural network is sometimes applied to solve combinatorial problems such as the traveling salesman problem, and used for layout design problems of printed boards. Since the Hopfield networks depend crucially on their nonlinear dynamics, in order to carrying out various proper operation, it is often requires to understand the existence of multiple equilibrium points or DC operating points. Therefore, it is important to have an efficient analysis method for obtaining a global picture of the dynamic behavior,

the equilibrium pattern and the basins of attraction in a given network. Here we show that the previous method can be used to complete such tasks.

Consider a N-neurons Hopfield network shown as Fig.2.16.

The behaviors can be described as a set of differential equations:

$$C_i \frac{du_i}{dt} = \sum_{j=1}^{N} w_{ij} x_j - \frac{u_i}{R_i} + I_i \tag{2.26.1}$$

$$x_i = f(u_i) = \frac{1}{2}(1 + tanh(\frac{u_i}{a})) \tag{2.26.2}$$

$$i = 1, 2, \ldots, N$$

where w_{ij} shows weight of synapse coupling, and it has such properties as $(a) w_{ij} = w_{ji}, (b) w_{ii} = 0$. Hence, the interconnect matrix W is a *symmetric matrix* whose diagonal elements are zeros. Based on above properties, we can conclude that the eigenvalues of W are all real by the matrix theories. It means that there only exist equilibrium points in Hopfield networks. The characteristics of the network largely rely on the sigmoid function described as (2.26.2). It may be abruptly changed as the parameter a tend to small. Hence the bifurcation phenomena would been occurred when a is varied. The equilibrium points can be found through solving follows equations:

$$\sum_{j=1}^{N} w_{ij} x_j - \frac{a}{2R_i} log \frac{x_i}{1 - x_i} + I_i = 0 \tag{2.27}$$

$$i \;=\; 1, 2, \ldots, N$$

Now we consider a practical Hopfield network used for layout problem of printed boards.[5]

[5]The example is given by Prof. A. Sakamoto at Tokushima university

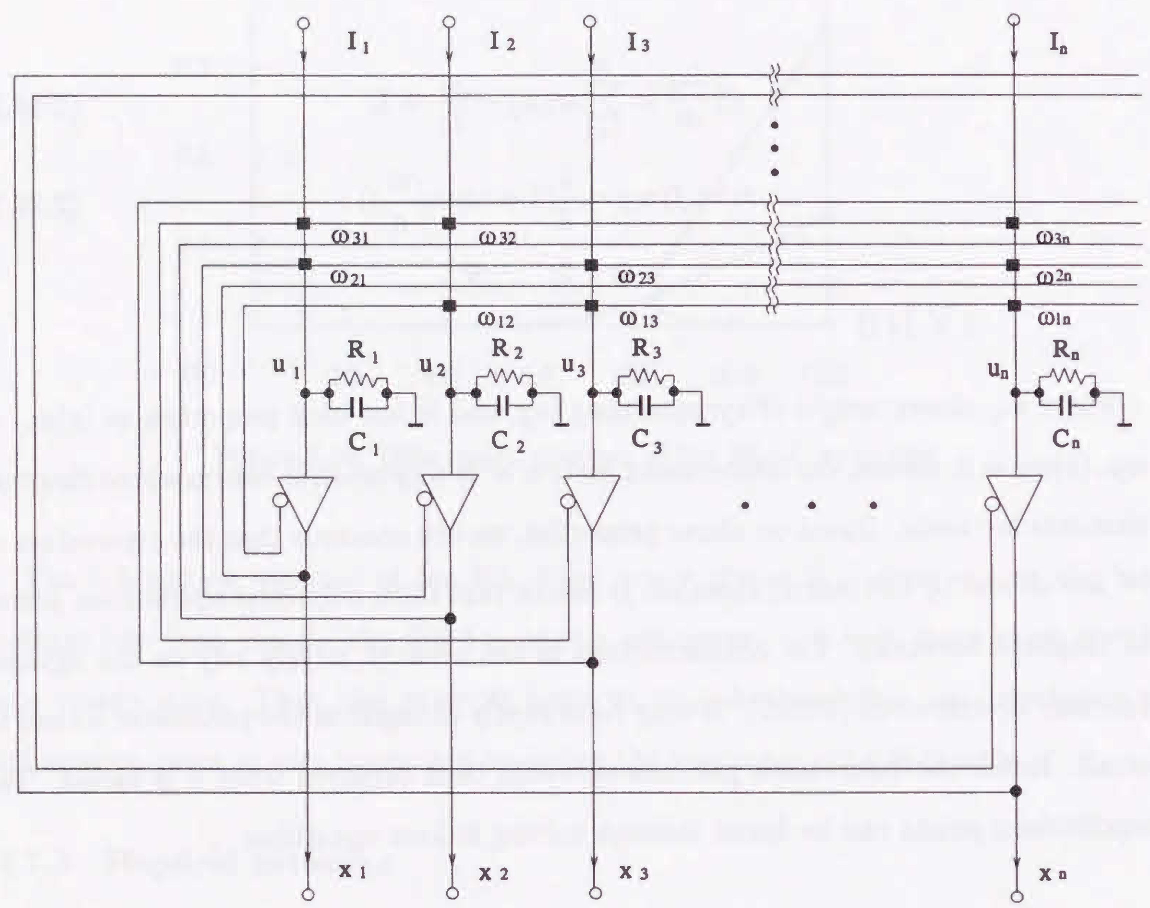


Figure 2.16: Hopfield neural network

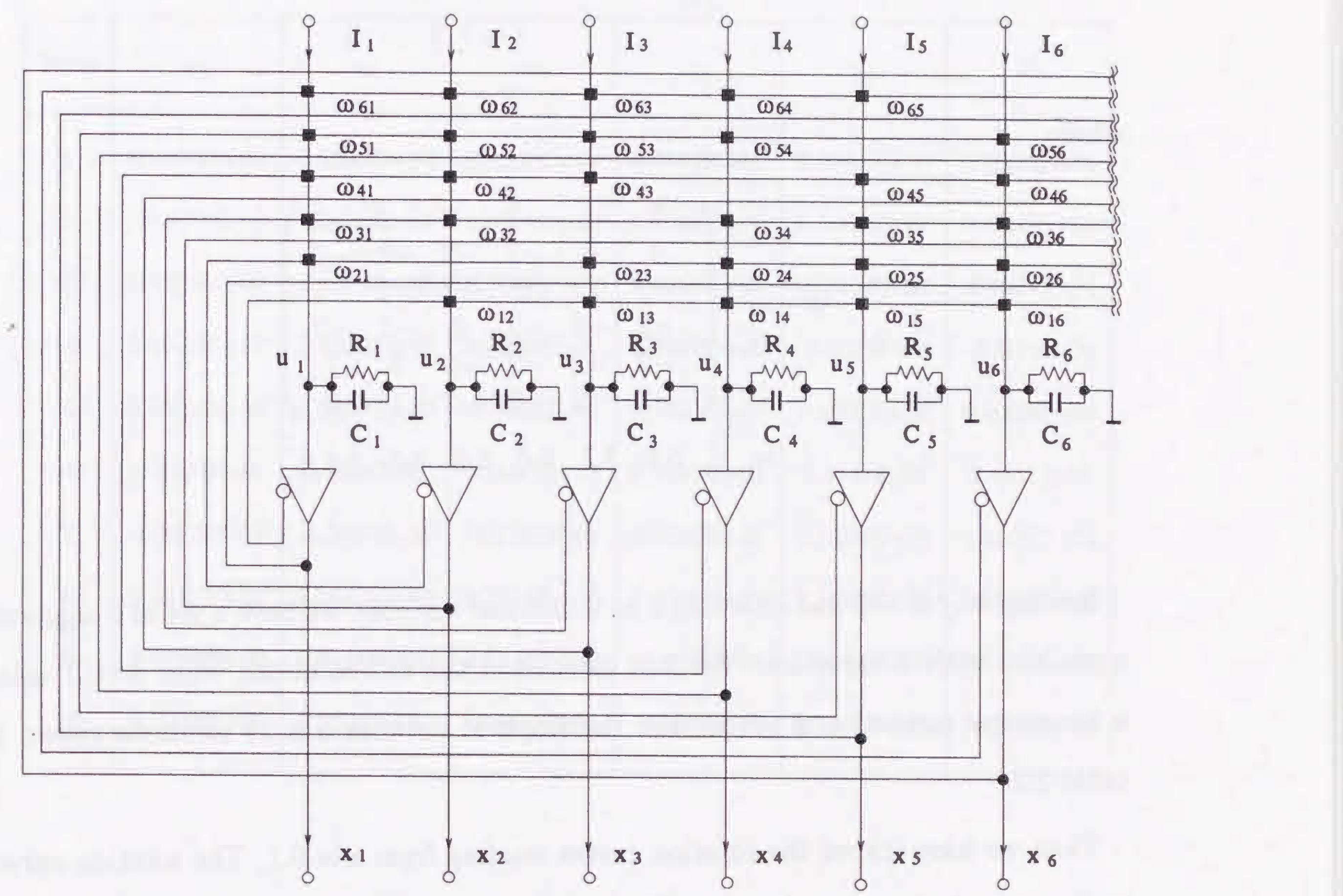


Figure 2.17: A Hopfield circuit having six neurons

The circuit contains 6 synapses whose equation is given by

$$\frac{du_i}{dt} = \sum_{j=1}^{6} w_{ij}x_j - \frac{a}{2}\log\frac{x_i}{1-x_i} + I_i$$
$$i = 1, 2, \ldots, 6$$

where

$$W = \begin{pmatrix} 0 & 1 & -2 & -2 & -2 & -2 \\ 1 & 0 & -2 & -2 & -2 & -2 \\ -2 & -2 & 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & -2 & 0 & 1 \\ -2 & -2 & -2 & -2 & 1 & 0 \end{pmatrix}$$
$$I = \begin{pmatrix} 3.5 & 3.5 & 5.0 & 5.0 & 3.5 & 3.5 \end{pmatrix}^T$$

Setting $du_i/dt = 0$ and choosing a as additional variable, we have a set of 6 algebraic equations with 7 variables. We first calculated the DC solutions when a=0.1 using a homotopy method and found that the Hopfield network has 19 solutions shown as table 2.2.

Then we have traced the solution curves starting from $a = 0.1$. The solution curves in (x_1, x_3, x_7)-plane are shown in Fig.2.18, where we choose $a = 0.29x_7 + 0.1$. There are 9 branching points and 4 limit points located by our curve tracing method shown as table 2.3. From the bifurcation diagram of the Hopfield network, we found that the characteristic of the Hopfield network is largely affected by the nonlinear controlling parameter a in the sigmoid function, there is only one solution for large a, and the number of solutions is increased as a decreased.

Table 2.2: The DC solutions of the Hopfield network for $a = 0.1$

Num	x_1	x_2	x_3	x_4	x_5	x_6
1	0.5000E+00	0.5000E+00	0.5000E+00	0.5000E+00	0.5000E+00	0.5000E+00
2	-0.3519E-04	-0.3519E-04	0.5000E+00	0.5000E+00	0.1000E+01	0.1000E+01
3	0.4999E+00	0.4999E+00	0.1000E+01	0.1000E+01	-0.3517E-04	-0.3517E-04
4	0.1000E+01	0.1000E+01	0.5000E+00	0.5000E+00	-0.3519E-04	-0.3519E-04
5	0.5000E+00	0.5000E+00	-0.5185E-05	0.1000E+01	0.5000E+00	0.5000E+00
6	0.4506E-04	0.4506E-04	0.1000E+01	0.1000E+01	0.4511E-04	0.4511E-04
7	-0.3519E-04	-0.3519E-04	-0.5185E-05	0.1000E+01	0.1000E+01	0.1000E+01
8	0.3842E+00	0.3842E+00	0.1854E+00	0.1000E+01	0.3842E+00	0.3842E+00
9	0.6158E+00	0.6158E+00	0.8146E+00	-0.1629E-04	0.6158E+00	0.6158E+00
10	0.5002E+00	0.5002E+00	-0.5199E-05	-0.5199E-05	0.1000E+01	0.1000E+01

(to be continued)

Num	x_1	x_2	x_3	x_4	x_5	x_6
11	0.5000E+00	0.5000E+00	0.1000E+01	-0.5185E-05	0.5000E+00	0.5000E+00
12	-0.3519E-04	-0.3519E-04	0.1000E+01	-0.5185E-05	0.1000E+01	0.1000E+01
13	0.3842E+00	0.3842E+00	0.1000E+01	0.1854E+00	0.3842E+00	0.3842E+00
14	-0.3517E-04	-0.3517E-04	0.1000E+01	0.1000E+01	0.4999E+00	0.4999E+00
15	0.6158E+00	0.6158E+00	-0.1629E-04	0.8146E+00	0.6158E+00	0.6158E+00
16	0.1000E+01	0.1000E+01	-0.5185E-05	0.1000E+01	-0.3519E-04	-0.3519E-04
17	0.1000E+01	0.1000E+01	-0.4518E-04	-0.4518E-04	0.1000E+01	0.1000E+01
18	0.1000E+01	0.1000E+01	0.1000E+01	-0.5185E-05	-0.3519E-04	-0.3519E-04
19	0.1000E+01	0.1000E+01	-0.5199E-05	-0.5199E-05	0.5002E+00	0.5002E+00

Table 2.3: Bifurcation points of the Hopfield network

BP	x_1	x_2	x_3	x_4	x_5	x_6	x_7
B_1	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.828E+01
B_2	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.516E+01
B_3	0.705E-02	0.705E-02	0.500E+00	0.500E+00	0.993E+00	0.993E+00	0.309E+01
B_4	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.500E+00	0.312E+01
B_5	0.993E+00	0.993E+00	0.500E+00	0.500E+00	0.705E-02	0.705E-02	0.312E+01
B_6	0.500E+00	0.500E+00	0.662E-01	0.934E+00	0.500E+00	0.500E+00	0.192E+01
B_7	0.500E+00	0.500E+00	0.937E+00	0.632E-01	0.500E+00	0.500E+00	0.189E+01
B_8	0.426E-01	0.426E-01	0.100E+01	0.100E+01	0.426E-01	0.426E-01	0.105E+01
B_9	0.960E+00	0.960E+00	0.116E-05	0.116E-05	0.960E+00	0.960E+00	0.101E+01
L_1	0.547E-03	0.547E-03	0.999E+00	0.999E+00	0.323E+00	0.323E+00	0.130E+01
L_2	0.324E+00	0.324E+00	0.999E+00	0.999E+00	0.543E-03	0.543E-03	0.130E+01
L_3	0.677E+00	0.677E+00	0.762E-03	0.762E-03	0.999E+00	0.999E+00	0.130E+01
L_4	0.999E+00	0.999E+00	0.780E-03	0.780E-03	0.676E+00	0.676E+00	0.130E+01

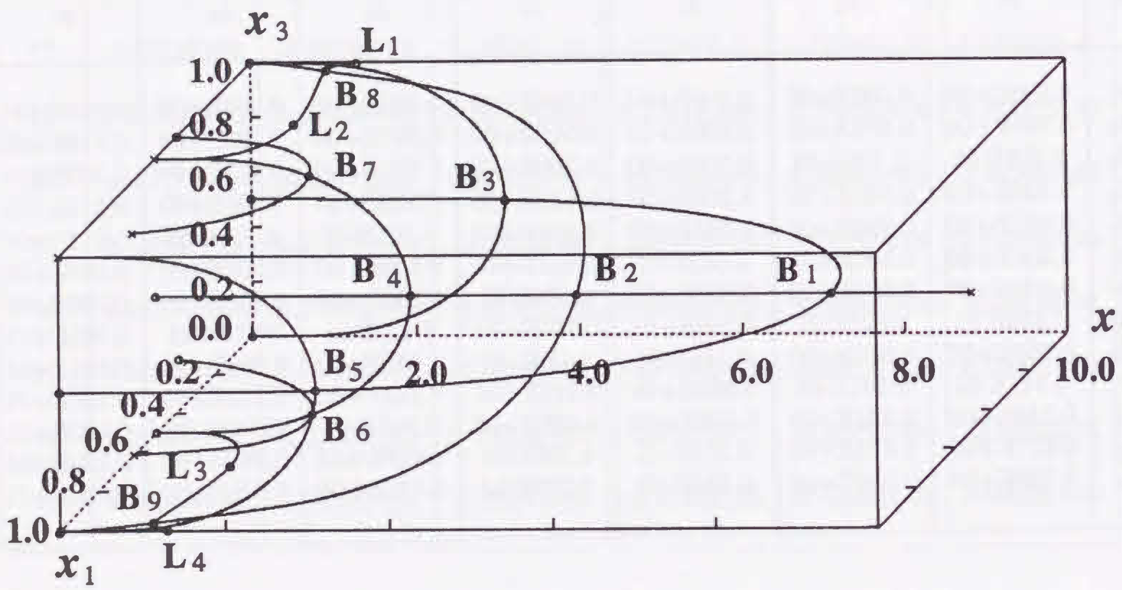


Figure 2.18: Bifurcation diagram of a Hopfield network

2.8 Conclusions and remarks

It is very important to calculate the bifurcation points to understand the behaviors of circuits. There are many kinds of bifurcation points such as limit point, branching point and isolated point. Since the rank of Jacobian matrix is decreased by one or more at these points, we cannot apply usual technique such as Newton-Raphson method to calculate them. In this chapter, we have shown that these points can be still found by the application of the Newton-Raphson method to the modified equation, which is termed as the direct method. As a indirect method, we consider a curve tracing method. First, we gave the description of our curve tracing method and investigate the properties of the curve nearby the bifurcation points when the curve tracing method is used to trace it. we proposed two theorems and pointed out that based on the theorems we can locate the bifurcation points on solution curves by simple sign test. We have also discussed the directions of branches at the branching points which are very useful for our curve tracing algorithm. A lot of examples were given, they shown that the characteristic curves of various circuits can be obtained by our curve tracing method embedded a variety of modification techniques proposed in this chapter.

References

[1] L.V.Kolev, "Find all solutions of nonlinear resistive equation via interval analysis, " *Int. J. Circuit Theory Appl.*, vol.12, pp.173-178, 1984.

[2] K.Yamamura and M.Ochiai, "An efficient algorithm for finding all solutions of piecewise-linear resistive circuits," *IEEE Trans. Circuits Syst.*, vol.CAS-39, pp.213-221, Mar. 1992.

[3] A.Ushida and L.O.Chua, "Tracing solution curve of nonlinear equations with sharp turning points," *Int. J. Circuit Theory Appl.*, vol.12, pp.1-21, Jan. 1984.

[4] M.Kubicek and M.Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures*, Springer-Verlag, 1983.

[5] C.B.Garcia and W.I.Zangwill, *Pathways to Solutions, Fixed Points, and Equilibria*, Prentice-Hall Englewood Cliff, N.J., 1981.

[6] L.O.Chua and P.M.Lin, *Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques*, Prentice-Hall Englewood Cliff, N.J., 1975.

[7] H.Kawakami, "Bifurcations of periodic responses in forced dynamical circuits: Computation of bifurcation values of system parameters," *IEEE Trans. Circuits Syst.*, vol.CAS-31, pp.248-260, Mar. 1984.

[8] N.Yamamoto, "Bifurcations of solutions of nonlinear equations involving parameters," *Theoretical and Applied Mechanics*, University of Tokyo press, vol.33, pp.435-444, 1985.

[9] H.B.Keller, *Numerical Methods in Bifurcation Problems*, Springer-Verlag, 1987.

[10] J.M.Ortega and W.C.Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, 1970.

[11] Y.Inoue, "Practical step length algorithms for tracing solution curves," *IEICE Trans.*, vol. J78-A, pp.381-390, Mar. 1995.

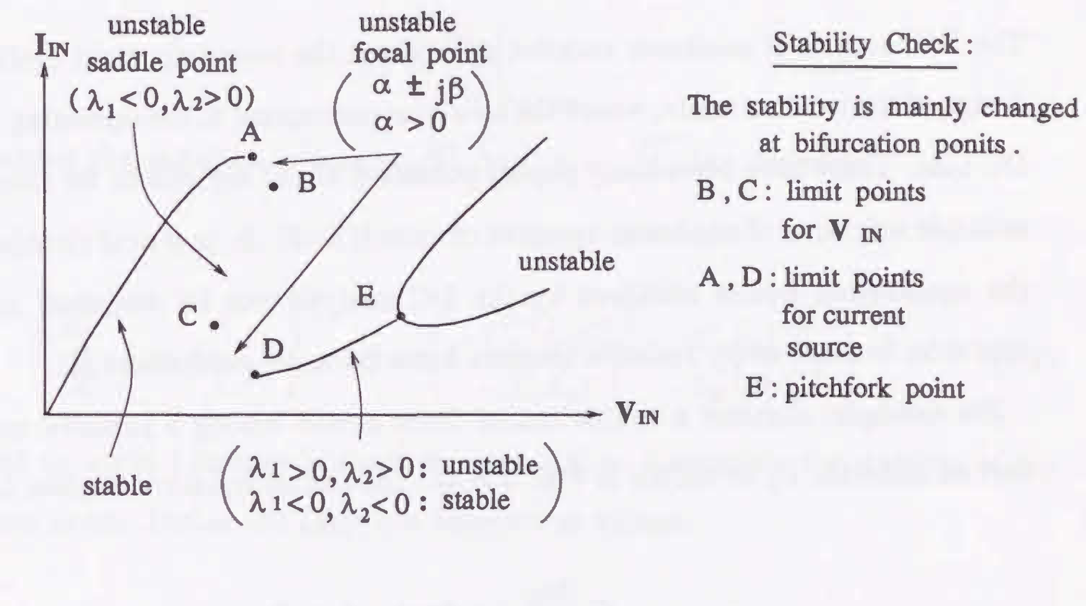
[12] K.Yamamura, "Simple algorithms for Tracing solution curves", *IEEE Trans.*, vol.40, pp.537-541, Aug. 1993.

Chapter 3

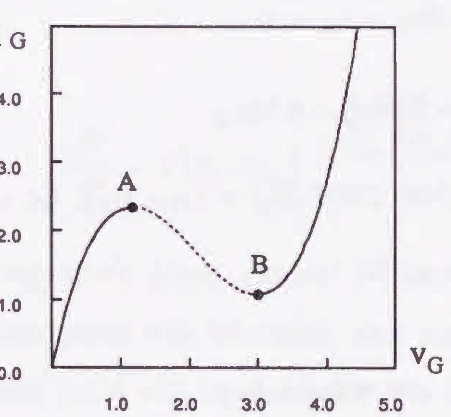
Stability of Characteristic Curve

3.1 Introduction

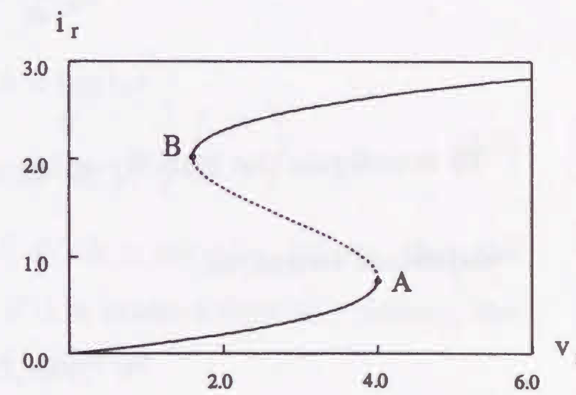
In this chapter, we discuss the stability of the characteristic curves for nonlinear resistive circuits including parasitic elements. Although the DC solution is determined by analyzing the nonlinear resistive circuit, its equilibrium point will be the one of stable or unstable because every resistive element has a small parasitic component in practice. We consider here two parasitic elements: a *capacitor* between every resistor and ground, and an *inductor* in series with each resistor. Of course, the stability can be decided by solving the variational equation at each equilibrium point obtained by the DC analysis, however that is very time-consuming. We show here that the stability is *mainly* changed at the boundary of the presence of negative differential resistance (NDR) and the bifurcation points such as turning and pitchfork points on the DC characteristic curves, so that the instability regions of the solution curve are easily found by both the locations of bifurcation points and NDR regions of the nonlinear resistors.



(a) Stability of characteristic solution curve



(b) NDR region of N-type nonlinear resistor



(c) NDR region of S-type nonlinear resistor

Figure 3.1: Stability change on solution curve for nonlinear resistive circuits

3.2 Derivation of stability problem

The DC analysis of nonlinear resistive networks is the most important problem for the design of electronic circuits, where the solutions correspond to the operating points for a DC bias. There have been many papers published about algorithms for calculating the multiple solutions of nonlinear resistive circuits [13-16]. In practical circuits, however, the equilibrium points obtained by the DC analysis can be described as stable or unstable, because every resistive element has a parasitic component [1].

For example, consider a simple tunnel diode circuit having a parasitic capacitor C_p and an inductor L_p as shown in Fig. 3.2(a). The circuit equation is given by

$$C_p \frac{dv_G}{dt} + i_G(v_G) - i_L = 0 \tag{3.1.1}$$

$$L_p \frac{di_L}{dt} + v_G + Ri_L - v_{in} = 0 \tag{3.1.2}$$

$$i_G(v_G) = 0.43v_G^3 - 2.69v_G^2 + 4.56v_G \tag{3.1.3}$$

To investigate the stability at an equilibrium point $E_Q = (v_{G0}, i_{L0})$, let us apply a variational technique.

$$v_G = v_{G0} + \Delta v_G, \quad i_L = i_{L0} + \Delta i_L \tag{3.2}$$

Then, we have

$$C_p \frac{d\Delta v_G}{dt} = -G_v \Delta v_G + \Delta i_L \tag{3.3.1}$$

$$L_p \frac{d\Delta i_L}{dt} = -\Delta v_G - R\Delta i_L \tag{3.3.2}$$

where

$$G_v = \left. \frac{\partial \hat{i}_G}{\partial v_G} \right|_{v_G = v_{G0}}$$

Transforming the variables into $x_1 = \sqrt{C_p}\Delta v_G$, $x_2 = \sqrt{L_p}\Delta i_L$, we have

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} -C_p^{-\frac{1}{2}} G_v C_p^{-\frac{1}{2}} & (C_p L_p)^{-\frac{1}{2}} \\ -(C_p L_p)^{-\frac{1}{2}} & -L_p^{-\frac{1}{2}} R L_p^{-\frac{1}{2}} \end{pmatrix} \tag{3.4}$$

Now, let us apply Liapunov's direct method [8,9] to investigate the stability of the equilibrium point. Define the Liapunov function as follows:

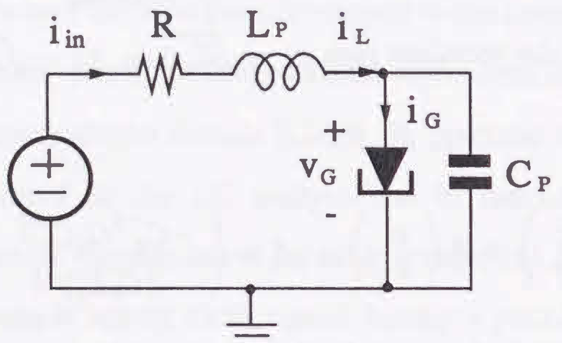
$$V = (x_1, x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.5}$$

then

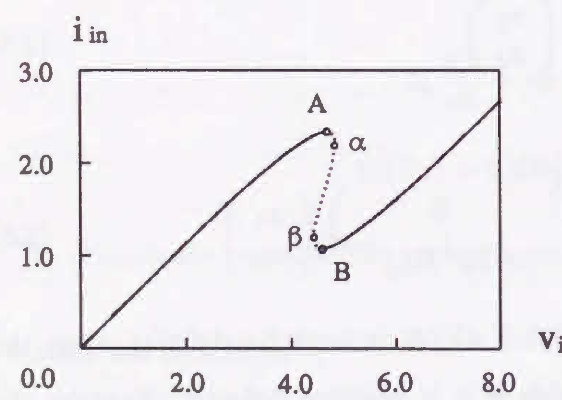
$$\frac{dV}{dt} = 2\,(x_1, x_2) \begin{pmatrix} -C_p^{-\frac{1}{2}} G_v C_p^{-\frac{1}{2}} & 0 \\ 0 & -L_p^{-\frac{1}{2}} R L_p^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.6}$$

The Liapunov's direct method [8] says that if dV/dt is negative definite, then the equilibrium point will be stable, and unstable if it is positive definite. Namely, the equilibrium point will be stable for $R > 0$ and $G_v > 0$.

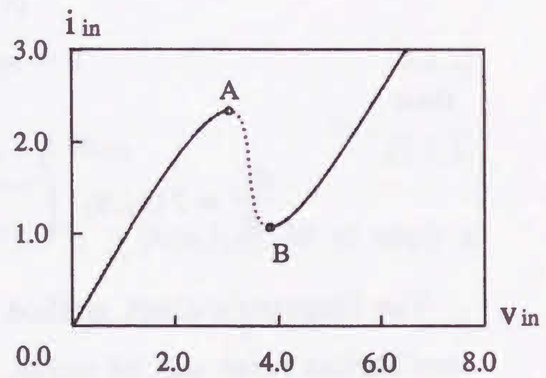
For $R > 0$ and $G_v < 0$, however, it is known that the equilibrium point is unstable if $tr(A) > 0$ in (3.4) is satisfied. On the other hand, the value is equal to the sum of diagonal elements of (3.6), and it is possible to become positive by appropriate selection of the parasitic elements. Hence, we conclude that for $R > 0$ and $G_v < 0$, the equilibrium point for some set of the parasitic elements becomes unstable in the sense



(a) Tunnel diode circuit



(b) Characteristic curve for large R



(c) Characteristic curve for small R

Figure 3.2: Driving point characteristic curves

of Liapunov, because the ratio can take

$$0 < L_p/C_p < \infty \tag{3.7}$$

even if the parasitic elements are small enough.

Now, consider the relationship in instability conditions between the Liapunov's direct method [8] and the analytical method. If L_p/C_p is large enough, the sign of dV/dt in (3.6) will be positive in the most part of the (x_1, x_2)-phase plane as shown in Fig. 3.3. In this case, although the equilibrium point $(0,0)$ is likely to be a saddle point from the behavior of trajectories, it may be an unstable focal, saddle or stable point depending on the ratios of L_p and C_p.

Now, we show the facts by the use of bifurcation diagram. Namely, the characteristic equation of (3.3) is given by

$$\lambda^2 + \left(\frac{G_v}{C_p} + \frac{R}{L_p}\right)\lambda + \frac{1}{C_p L_p}(RG_v + 1) = 0$$

Hence, the bifurcation diagram in (R, G_v) plane is shown in Fig. 3.4 for $C_p/L_p = 0.2$, where the equilibrium point is classified as follows :

$$\text{(saddle point)}, \quad \text{for } RG_v + 1 < 0 \tag{3.8.1}$$

$$\text{(unstable focal point)}, \quad \text{for } \frac{G_v}{C_p} + \frac{R}{L_p} < 0 \tag{3.8.2}$$

Observe that, for $R > 0$ and $G_v < 0$, the instability region is increased as the ratio C_p/L_p is decreased, and whole of the domain satisfying $G_v < 0$ will be unstable for

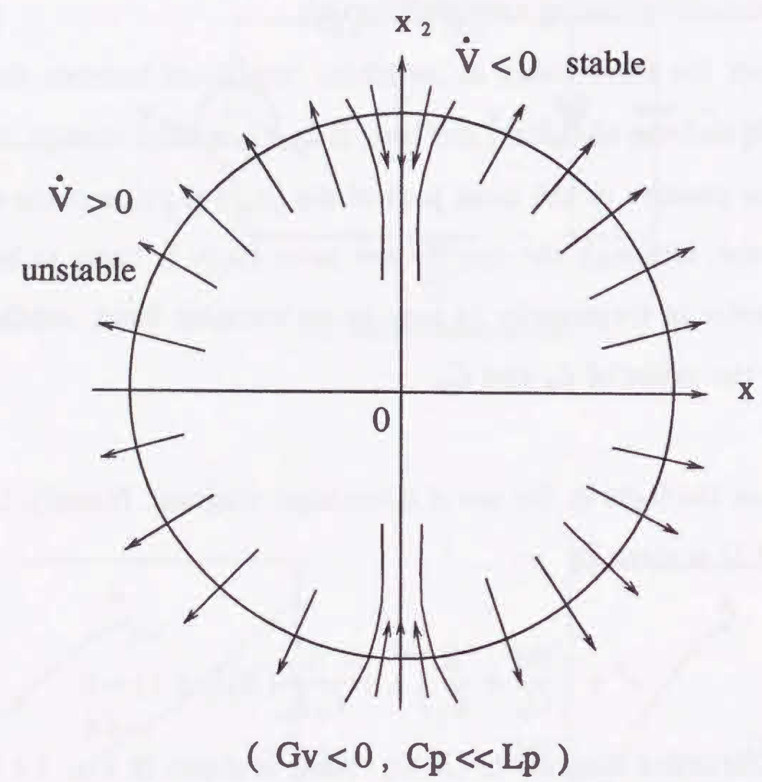


Figure 3.3: Liapunov's (x_1, x_2) phase plane

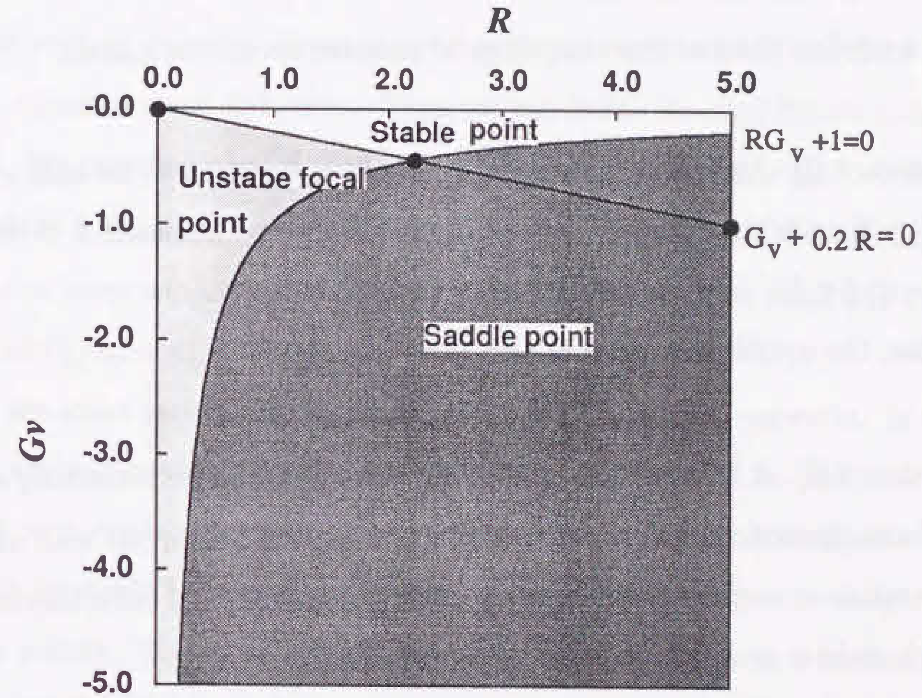


Figure 3.4: Bifurcation diagram in (R, G_v) plane

$C_p/L_p \longrightarrow 0$. The result is the same as the Liapunov's direct method (3.6) saying that if one or more diagonal elements has positive value, it is possible to become unstable by appropriate selection of the parasitic elements. Note that this criterion corresponds to the worst case of stabilities, in the sense that the equilibrium point becomes unstable for some set of the parasitic elements and otherwise, it is stable.

Now, we define the stability conditions of resistive circuits as follows.

Definition 1 [3]: An equilibrium point x^* is said to be *stable* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $\| x(t) - x^* \| < \epsilon$ for all $t \geq t_0$, whenever $\delta > 0$ such that $\| x(t_0) - x^* \| < \delta$.
Otherwise, the equilibrium point is said to be *unstable.*

Definition 2 [3]: A DC circuit's equilibrium point is said to be *potentially stable* if the point of any dynamic circuit created by augmenting the DC circuit with an arbitrary set of parasitic capacitors between every node and ground, and inductors in the series with each resistor is stable.

On the other hand, we define instability conditions as follows:
Definition 3: A resistive circuit's equilibrium point is said to be *k-th order saddle-node* unstable if the characteristic equation obtained from the variational dynamic circuit has k positive real roots. An equilibrium point is said to be *occasionally* unstable if the above dynamic circuit is unstable only for some set of the parasitic capacitors and inductors, and is stable for some other parasitic elements. [1]

[1] Note that the saddle-node equilibrium point corresponds to both *open-* and *short-circuit unstable* points in Ref.[21-22]. The occasionally unstable points correspond to the *open-circuit unstable* points.

Definition 4: For reciprocal nonlinear circuits, an equilibrium point is said to be *negative differential resistance (NDR) unstable* if, for a resistive circuit with parasitic elements, dV/dt has one or more positive real diagonal elements.

Note that if it satisfies $tr(A) > 0$ for the variational equation such as (3.4), we will have *(the sum of diagonal elements)* > 0 in dV/dt such as (3.6) for some set of the parasitic capacitors and inductors. Thus, we can define the equilibrium point is NDR unstable. We can show that the *NDR unstable* condition contains the two instabilities in *Definition 3*, because the *occasionally* and *saddle-node* instabilities will happen only when one or more nonlinear differential resistances has negative value (see Fig. 3.4 for the case of $C_p/L_p = 0$).

There are some papers discussing the stability of nonlinear networks. In references [2,10], a globally asymptotically stable condition is discussed for nonlinear dynamic networks in a qualitative manner. In reference [3], a simple technique is proposed to identify unstable DC operating points, where the instability is investigated by the Jacobian matrix. The method gives the same result as *Theorem 2* in this paper. In reference [21,22], they apply it to the stability of driving-point characteristic curves for nonlinear one-port resistive circuits.

In this study, we discuss how to identify the unstable regions of solution curves such as driving-point and transfer characteristic curves, where the parasitic elements can take the region of (3.7). We find that if one or more differential resistances has negative value, those parts of solution curve will be unstable. Furthermore, at the bifurcation points such as the turning and branch points [4 -5], stability of the equilibrium point is changed from a some kind of stabilities to a saddle-nodal point. So, unstable regions of the solution curve can be easily identified without investigating the variational equation. Thus, the technique is very useful for identifying the unstable regions of solution curves.

3.3 Stability of nonlinear reciprocal resistive circuits

3.3.1 Dynamic equation for nonlinear resistive circuit considering parasitic elements

Now, consider a nonlinear reciprocal resistive circuit having both voltage- and current-controlled nonlinear resistors. To obtain the DC circuit equation, partition the circuit into two groups as follows:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad i = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \tag{3.9.1}$$

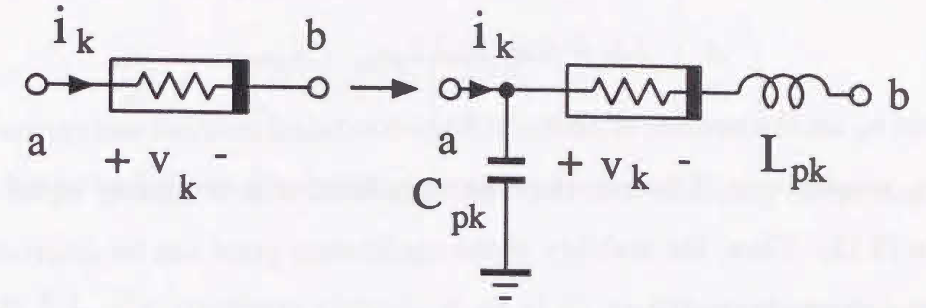
$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad A = (A_1 A_2) \tag{3.9.2}$$

where A is the incidence matrix, and the subscript "1" indicates the elements of voltage-controlled resistors (including linear resistors), and "2" those of current-controlled resistors, respectively. Then, we have the following circuit equation using the *tableau approach* [6].

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -A_1^T \\ 0 & 0 & 0 & 1 & -A_2^T \\ A_1 & A_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ v_1 \\ v_2 \\ v_n \end{pmatrix} - \begin{pmatrix} g_v(v_1) \\ r_i(i_2) \\ E_1 \\ E_2 \\ AJ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.10}$$

where v_n denotes the node voltage. From the first and third rows, we have

$$i_1 = g_v(A_1^T v_n + E_1) \tag{3.11}$$



(a) A nonlinear resistor (b) Insertion o f parasitic capacitor and inductor

Figure 3.5: A schematic diagram of insertion of parasitic capacitor and inductor

Thus, we have the following relations from the second and fourth rows in (3.10)

$$A_2 i_2 = -A_1 g_v(A_1^T v_n + E_1) + AJ \tag{3.12.1}$$

$$A_2^T v_n + E_2 = r_i(i_2) \tag{3.12.2}$$

The first equation corresponds to the *nodal equation*, and the second one is obtained by the *Kirchhoff's voltage law* for the loops containing the current-controlled resistors.

Now, let us derive the dynamic equation by considering the parasitic elements. We assume a parasitic capacitor between every resistor and ground, and a parasitic inductor in series with each resistor as shown in Fig. 3.5. Then, we have

$$C_p \frac{dv_n}{dt} = -A_2 i_2 - A_1 g_v(v_1) + AJ \tag{3.13.1}$$

$$L_{p1} \frac{dg_v(v_1)}{dt} = -v_1 + A_1^T v_n + E_1 \tag{3.13.2}$$

$$L_{p2} \frac{di_2}{dt} = A_2^T v_n + E_2 - r_i(i_2) \tag{3.13.3}$$

where

$$C_p = \mathrm{diag}(C_{p1}, C_{p2}, \ldots, C_{pN})$$

$$L_{p1} = \mathrm{diag}(L_{pv1}, L_{pv2}, \ldots, L_{pvn_1})$$

$$L_{p2} = \mathrm{diag}(L_{pi1}, L_{pi2}, \ldots, L_{pin_2})$$

N, n_1 and n_2 are the number of nodes, voltage-controlled resistors and current-controlled resistors, respectively. Observe that the right-hand side is exactly equal to the DC equation (3.12). Thus, the stability of the equilibrium point can be determined by the variational dynamic equation.

3.3.2 Stability criterion for nonlinear reciprocal resistive circuits

Assume the equilibrium point be (v_{n0}, v_{10}, i_{20}), and put

$$v_n = v_{n0} + \Delta v_n, \quad v_1 = v_{10} + \Delta v_1 \quad i_2 = i_{20} + \Delta i_2 \tag{3.14}$$

We further transform the variables as follows:

$$x_1 = \sqrt{C_p}\Delta v_n, \quad x_2 = \sqrt{L_{p1}}G_v\Delta v_1 \quad x_2 = \sqrt{L_{p2}}\Delta i_2 \tag{3.15}$$

Then, we have from (3.13)

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = P \begin{pmatrix} 0 & -A_1G_v & -A_2 \\ G_vA_1^T & -G_v & 0 \\ A_2^T & 0 & -R_i \end{pmatrix} P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.16}$$

where $P = \mathrm{diag}(C_p^{-\frac{1}{2}}, L_{p1}^{-\frac{1}{2}}G_v^{-1}, L_{p2}^{-\frac{1}{2}})$, and

$$G_v = \left.\frac{\partial g_v(v)}{\partial v}\right|_{v=v_{10}}, \quad R_i = \left.\frac{\partial r_i(i)}{\partial i}\right|_{i=i_{20}}$$

Now, let us define the Liapunov function as follows:

$$V = \left(x_1^T x_2^T x_3^T\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.17}$$

Then, we have

$$\frac{dV}{dt} = -2\left(x_1^T x_2^T x_3^T\right) P \begin{pmatrix} 0 & 0 & 0 \\ 0 & G_v & 0 \\ 0 & 0 & R_i \end{pmatrix} P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.18}$$

Now, we have the following instability condition for the nonlinear reciprocal re sistive circuits.

Theorem 3.1: *For reciprocal circuits, if all of G_v and R_i have positive differential resistances at the equilibrium point, those parts of the characteristic curve will be potentially stable. On the other hand, the equilibrium point will be NDR unstable if one or more of the differential resistances has negative value at the point.*

Liapunov's direct method [8] says that if the relation (3.18) is negative definite, the equilibrium point will be stable. However, if it has negative differential resistances, the equilibrium point will be unstable in the sense of *Definition 4*. Furthermore, it might be a *saddle-node* or *occasionally* unst able point depending on the ratios of parasitic capacitors and inductors as show n in Fig. 3.4.

Example 3.1 Consider a current-controlled resistive circuit as shown in Fig. 3.6. The circuit equation is given by follows:

$$C_p\frac{dv_C}{dt} = i_{L1} - i_{L2}$$

$$L_{p1}\frac{di_{L1}}{dt} = v_{in} - v_C - v_r(i_{L1})$$

$$L_{p2}\frac{di_{L2}}{dt} = v_C - Ri_{L2}$$

$$V_r(i_r) = 0.43i_r^3 - 2.69i_r^2 + 4.56i_r$$

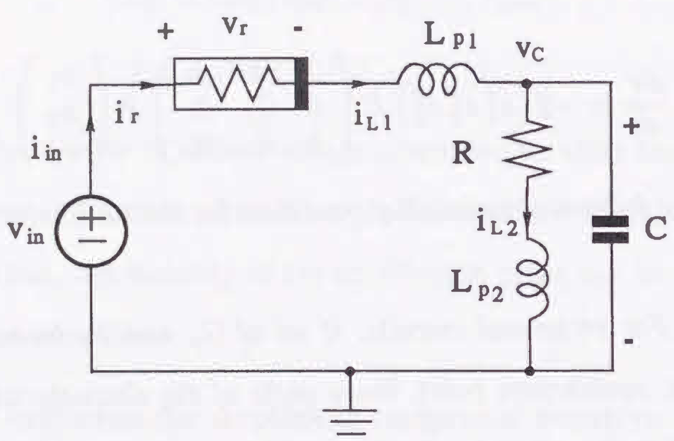


Figure 3.6: Current-controlled resistive circuit with parasitic elements

where $R = 3.5, L_p1 = 1., L_p2 = 1., C_p = 0.5$.

The characteristic curve of nonlinear resistor and the driving point characteristic curve of the circuit as shown in Fig. 3.7.

Here we show the unstable region by dotted line. Although the derivative is always $di_{in}/dv_{in} > 0$, the equilibrium point in the region (α, β) corresponds to unstable focal point for the given parameters, where the stability is checked by the eigenvalues of the variational dynamic equation. We found that, increasing the ratio L_{p1}/C_p, the unstable region (α, β) approaches to (A, B) corresponding to the negative resistance region (A, B) in Fig. 3.7(a). Thus, the region (A, B) in Fig. 3.7(b) corresponds to the *occasionally unstable* equilibrium point.

3.4 Stability criterion by curve tracing method

Now, consider more general circuits containing bipolar transistors, FETs and so on. The solution curve can be calculated by solving a resistive circuit composed of n equa-

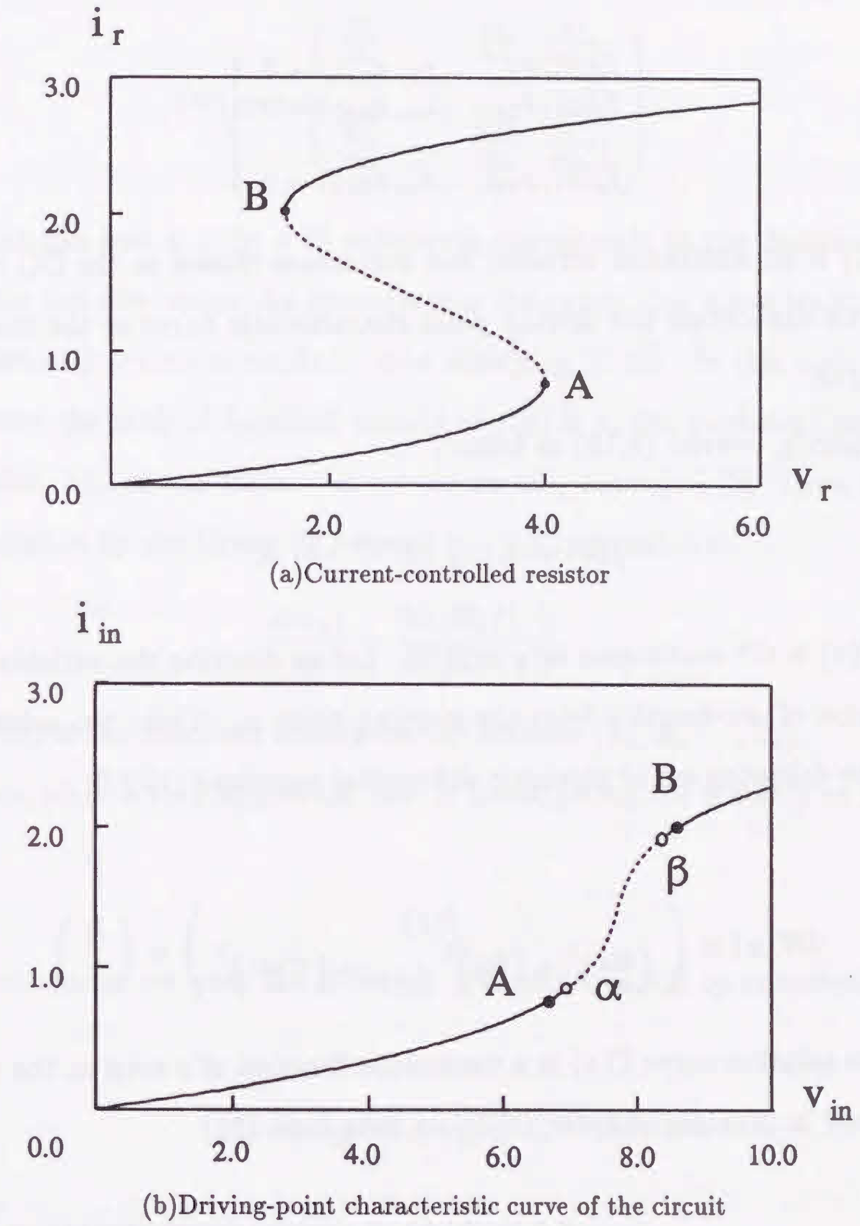


Figure 3.7: Stability of the DP characteristic

tions in $(n+1)$ variables

$$\left.\begin{array}{l} f_1(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ f_2(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n(x_1, x_2, \ldots, x_n, x_{n+1}) = 0 \end{array}\right\} \tag{3.19}$$

where x_{n+1} is an additional variable, and sometimes chosen as the DC bias or forced input, so we can obtain the driving point characteristic curve or the transfer characteristic curve.

For simplicity, rewrite (3.19) as follows:

$$f(x) = 0, \quad f : R^{n+1} \longrightarrow R^n \tag{3.20}$$

Assume $f(x)$ is C^2 continuous in $x \in R^{n+1}$. Let us describe the variable by $x = x(s)$ as a function of *arc-length* s from the starting point x_0. Then, the solution of (3.20) satisfies the following set of algebraic-differential equations [15,17]:

$$D\Gamma(x) \equiv \begin{pmatrix} f(x) \\ \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \cdots + \left(\frac{dx_{n+1}}{ds}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.21}$$

Since the solution curve $\Gamma(x)$ is a continuous function of s even at the *turning point* as we proved in previous chapter[15][5], we have from (21)

$$D\Gamma(x) \begin{pmatrix} \frac{dx_1}{ds} \\ \cdots \\ \frac{dx_n}{ds} \\ \frac{dx_{n+1}}{ds} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3.22}$$

where

$$D\Gamma(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \\ \frac{dx_1}{ds} & \cdots & \frac{dx_n}{ds} & \frac{dx_{n+1}}{ds} \end{pmatrix}$$

Observe that the first $n \times (n+1)$ submatrix corresponds to the Jacobian matrix of $f(x)$, and the last row shows the derivatives of the curve. Our curve tracing algorithm [15] can efficiently trace the solution curve satisfying (3.22) . In this case, it is proved that whenever the rank of Jacobian matrix of $f(x)$ is n, the coefficient matrix $D\Gamma(x)$ is nonsingular. So, we can trace even for the *turning points*[15] [5]. Thus, we have the following relation by the Cramer's formula to $(n+1)$th variable

$$\frac{dx_{n+1}}{ds} = \frac{\det|D_n f(x)|}{\det|D\Gamma(x)|} \tag{3.23}$$

where $D_n f(x)$ is the Jacobian matrix for the variable $\{x_1, x_2, \ldots, x_n\}$.

This relation plays a very important role in investigating the stability of the solution curves.

Now, assume that we have the following dynamic equation by considering parasitic elements:

$$Q\frac{dx}{dt} = f(x) \tag{3.24}$$

where Q is a diagonal matrix similar to (3.13) as follows:

$$Q = \begin{pmatrix} C_p & 0 \\ 0 & L_p \end{pmatrix}$$

Then, the variational equation at an equilibrium point x is given by

$$Q\frac{d\Delta x}{dt} = D_n f(x)\Delta x \tag{3.25}$$

Thus, the stability condition of the resistive circuits is decided by the eigenvalues of the Jacobian matrix $D_n f(x)$. We have the following stability property around the *turning point.*

Theorem 3.2 *When the solution curve has passed through the turning point, one of three bifurcations of the stability may occur at the point as follows: a potentially stable to a saddle-node, an unstable focal to a saddle-node, and a saddle-node to a different order of saddle-node.*

Proof. The direction of the solution curve is changed at a turning point, so that we have $dx_{n+1}/ds = 0$ at the point. This means that the sign of $\det|D_n f(x)|$ is changed after passing through the turning point because of the nonsingularity of $D\Gamma(x)$ in (3.23) [5]. Here, we transform (3.25) as follows:

$$\frac{d\Delta x}{dt} = Q^{-1}D_n f(x)\Delta x$$

The eigenvalues of the variational equation satisfy the following relation [3,7] :

$$\det|Q^{-1}D_n f(x)| = \det|Q^{-1}|\prod_{i=1}^{n}\lambda_i \tag{3.26}$$

We assume that $\det|Q^{-1}| \neq 0$ holds, so that the stability depends on the eigenvalues of $D_n f(x)$, where $\lambda_i (i = 1, 2, \ldots, n)$ are the eigenvalues composed of real and/or complex conjugates. Thus, the change of sign (3.23) means that an odd number of the real eigenvalues has changed signs after passing through the turning point. Therefore,

the type of stability is changed at the point. Note that for the complex conjugate eigenvalues, the sign of (3.26) does not change even if the real parts have changed sign, so that one of three bifurcations given in the *Theorem 3.2* will be possible.

□

Suppose that $dx_{n+1}/ds > 0$ holds at the starting point of the solution curve and the equilibrium point is stable. Then, the regions $dx_{n+1}/ds < 0$ of the solution curve are unstable, where it has an odd number of positive real eigenvalues so that the equilibrium point is a saddle-nodal type unstable point [18].

Remark: We assume again that the starting point is stable and $dx_{n+1}/ds > 0$. Then, there are two cases of the stability in the region of $dx_{n+1}/ds > 0$. The first, all of the real eigenvalues are negative where the region is st able. The second, it has an even number of positive real eigenvalues where the region is unstable.

Next, we consider the stability of the solution curve around the *branch bifurcation point* [4], where two solution curves cross at a point. It is known that the rank of the Jacobian matrix of (3.19) for $\{x_1, x_2, \ldots, x_{n+1}\}$

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \end{pmatrix}$$

is reduced to $n-1$ or less than $n-1$. Hence, the matrix $D\Gamma(x)$ becomes singular at the bifurcation point. We have the following theorem around the point.

Theorem 3.1 *Let $\Gamma(x)$ be a smooth solution curve passing through the branching bifurcation point. Then, the stability of solution is changed at the point.*

Proof: For simplicity, put

$$d_1(x_{n+1}) \equiv \det|D\Gamma(x)| \tag{3.27}$$

Now, applying Taylor expansion to $d_1(x_{n+1})$ at two points $x^*_{n+1} - \Delta x_{n+1}$ and $x^*_{n+1} + \Delta x_{n+1}$ before and after the bifurcation point x^*, we have

$$d_1(x^*_{n+1} - \Delta x_{n+1}) = d_1(x^*_{n+1}) - d_1'(x^*_{n+1})\Delta x_{n+1} + \cdots \tag{3.28.1}$$

$$d_1(x^*_{n+1} + \Delta x_{n+1}) = d_1(x^*_{n+1}) + d_1'(x^*_{n+1})\Delta x_{n+1} + \cdots \tag{3.28.2}$$

where $'$ indicates the derivative with respect to x_{n+1}. At the branching point x^*, the following relations hold [4]

$$\mathrm{rank}(Df(x^*)) = n-1, \quad d_1(x^*_{n+1}) = 0, \quad d_1'(x^*_{n+1}) \neq 0 \tag{3.29}$$

Multiplying the two equations in (3.28), we obtain

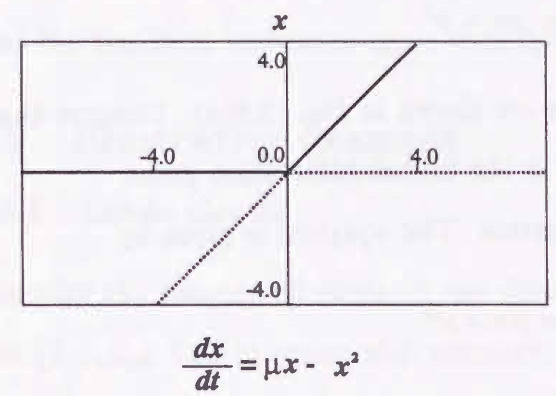
$$d_1(x^*_{n+1} - \Delta x_{n+1}) d_1(x^*_{n+1} + \Delta x_{n+1}) \cong -[d_1'(x^*_{n+1})]^2 \Delta x^2_{n+1}$$

Thus, the sign of the denominator of (3.23) is changed whenever it passes through the point. We have the same result as $\det|D\Gamma(x)|$ for

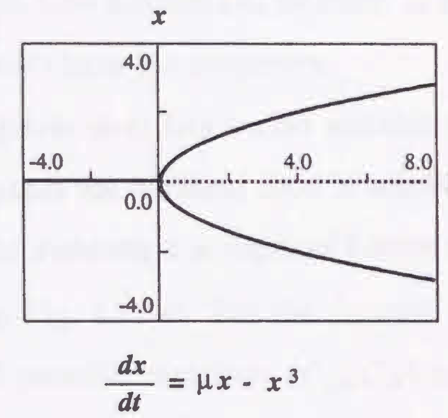
$$d_2(x_{n+1}) \equiv \det|D_n f(x)| \tag{3.30}$$

because the rank of $Df(x)$ is $n-1$ at the bifurcation point.

Thus, both signs of the denominator and numerator of (3.23) are changed at the point, so that the direction of solution curve dx_{n+1}/ds will not be changed at the branch bifurcation point. But the stability of the solution curve is changed. The instability of



(a) Branch point



(b) Pitchfork point

Figure 3.8: Illustrative examples of a branch bifurcation and pitchfork bifurcation

the equilibrium point after the bifurcation point will be a saddle type.

□

As a special case, there are many symmetric circuits such as Flip-Flop circuit. In this case, they sometimes have an interesting property such as one of the solution curves is *symmetric* with respect to another one as shown in Fig. 3.8(b). This type of bifurcation is termed as a *pitchfork* bifurcation [4,12].

Corollary 1 *At a pitchfork point, one of the solution curves changes stability at the point, while the other keeps the same stability passing through the point.*

Example 3.2 Now, consider two simple examples shown in Fig. 3.8(a) and (b). The first one is a branch bifurcation. The equation is given by

$$\frac{dx}{dt} = \mu x - x^2$$

The solution curves and their stabilities are shown in Fig. 3.8(a). Observe that the stabilities of both branches are changed at the branch bifurcation point.

The second example is a pitchfork bifurcation. The equation is given by

$$\frac{dx}{dt} = \mu x - x^3$$

The solution curves and their stabilities are shown in Fig. 3.8(b). Although the stability of the solution curve $x = 0$ is changed at $\mu = 0$, $\mu = x^2$ has the same stability before and after the pitchfork point.

Remark: In *Theorem 3.2* and *3.3*, the instability regions are determined by investigating whether the variational equation has positive real eigenvalues or not. However, it may sometimes happen that they have complex conjugate eigenvalues having positive real parts. For this kind of instability, the equilibrium point is an *unstable focal point* [18], and the sign of dx_{n+1}/ds is not changed at the bifurcation point. This bifurcation is called *Hopf bifurcation* [20].

Example 3.3 Consider the circuit given in Fig. 3.2(a). It has different kinds of characteristic curves as shown in Fig. 3.2(b) and (c) depending on the value of resistor R. In Fig. 3.2(b), we have a region (α, β) of $dv_{in}/ds < 0$, so that the equilibrium points in the region are *saddle-node points.* On the other hand, we have the unstable regions of $dv_{in}/ds > 0$ such as (A, α) , (β, B) in Fig. 3.2(b) and (A, B) in Fig. 3.2(c) corresponding to the NDR region. The transient waveforms starting from the vicinity of each equilibrium point are shown in Fig. 3.9 and Fig. 3.10, respectively. We can see a short period of small oscillation which grows larger and then settles down to the

stable DC solution. On the other hand, the unstable equilibrium point in (A, B) gives rise to a limit cycle, because the region does not have a stable DC solution. It is clear that the transient responses agree with the results from the properties.

3.5 Illustrative example

3.5.1 Diode circuit

Consider the two-tunnel-diode circuit shown in Fig. 3.11(a). Put the parasitic inductors (L_{p1}, L_{p2}, L_{p3}) in series with resistors, and parasitic capacitors (C_{p1}, C_{p2}) between every node and ground. Thus, we have the dynamic circuit shown in Fig. 3.11(b).

The circuit equation is given by

$$C_{p1}\frac{dv_{Cp1}}{dt} = i - g_1(v_1)$$

$$C_{p2}\frac{dv_{Cp2}}{dt} = g_1(v_1) - g_2(v_2)$$

$$L_{p1}\frac{di}{dt} = v_{in} - v_{Cp1} - Ri$$

$$L_{p2}\frac{dg_1(v_1)}{dt} = -v_1 + v_{Cp1} - v_{Cp2}$$

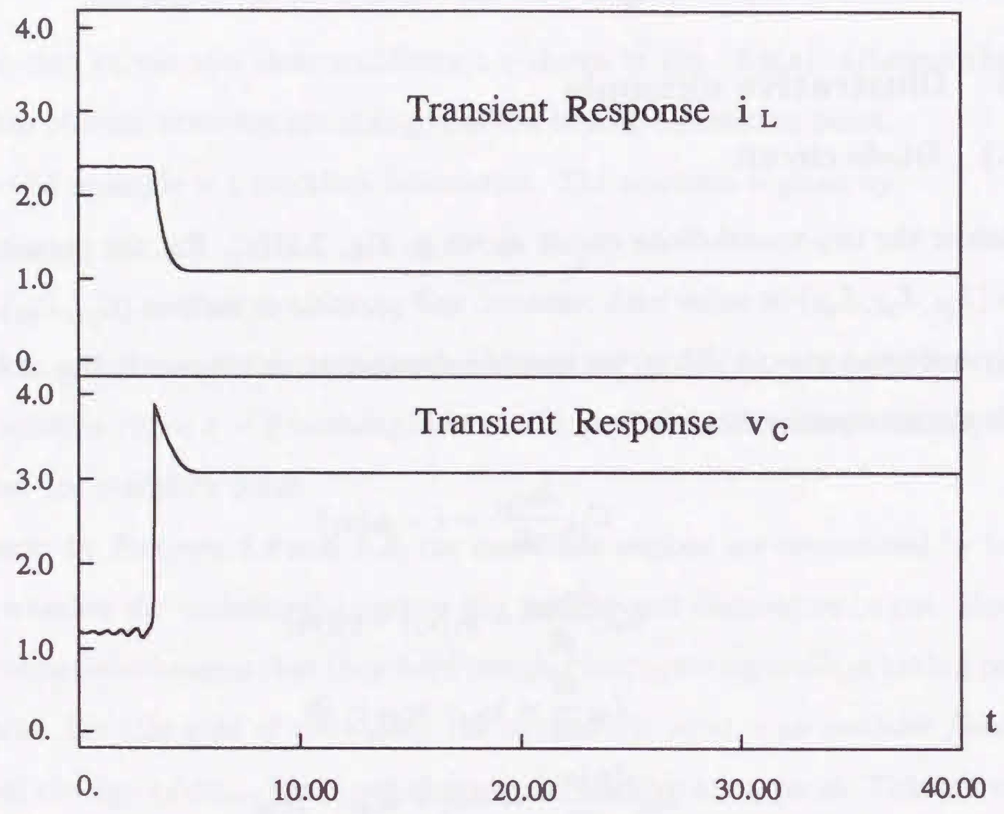
$$L_{p3}\frac{dg_2(v_2)}{dt} = -v_2 + v_{Cp2}$$

where

$$g_1(v_1) \equiv 2.5v_1^3 - 10.5v_1^2 + 11.8v_1$$

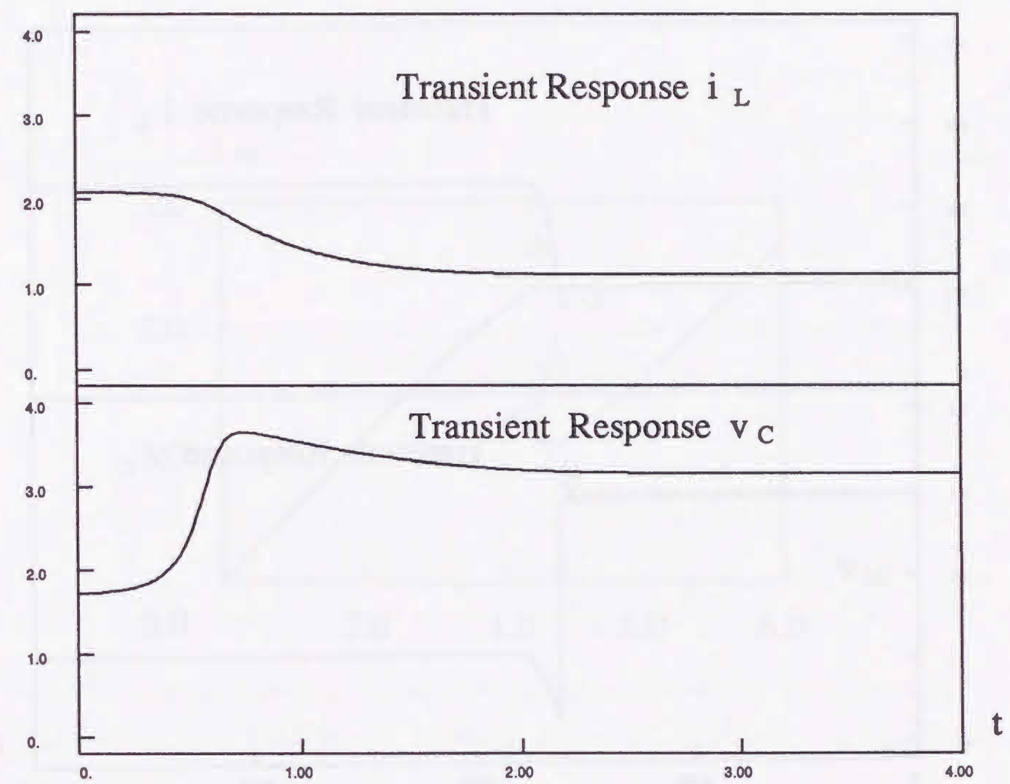
$$g_2(v_2) \equiv 0.43v_2^3 - 2.69v_2^2 + 4.56v_2$$

We got the transfer characteristic curve shown as Fig. 3.12 by our curve tracing method for $R = 1.5$ and shown the unstable regions of curves by dotted lines.



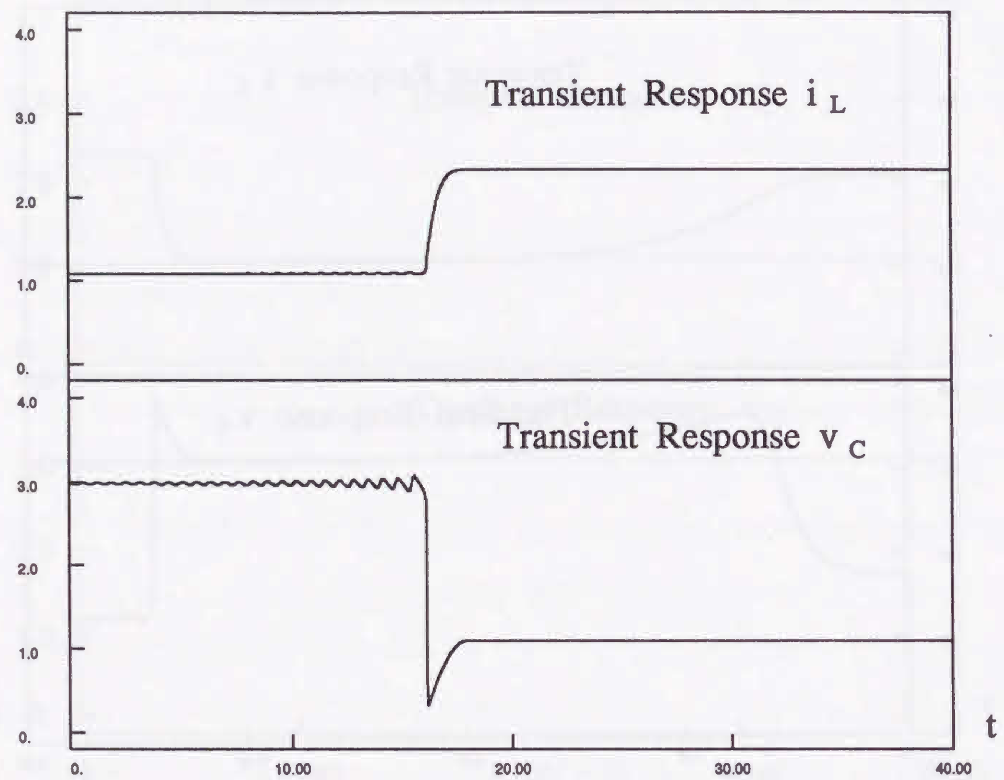
(a) Equilibrium point between (A,α)

(to be continued)



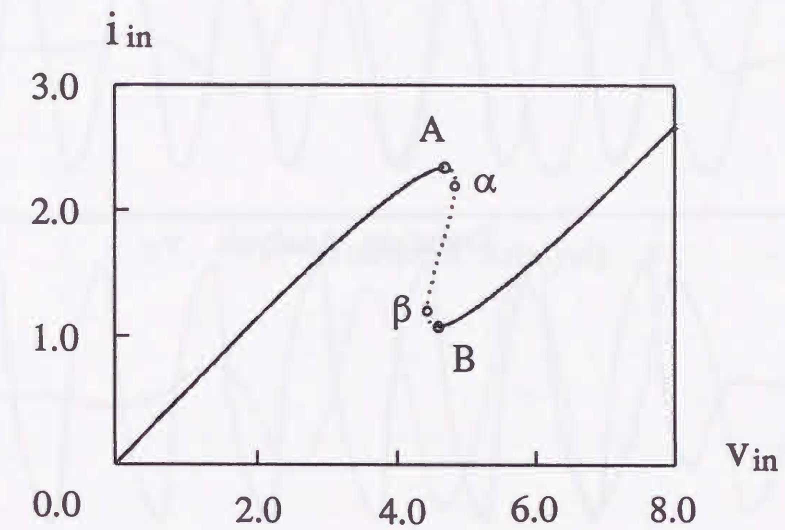
(b) Equilibrium point between (α, β)

(to be continued)



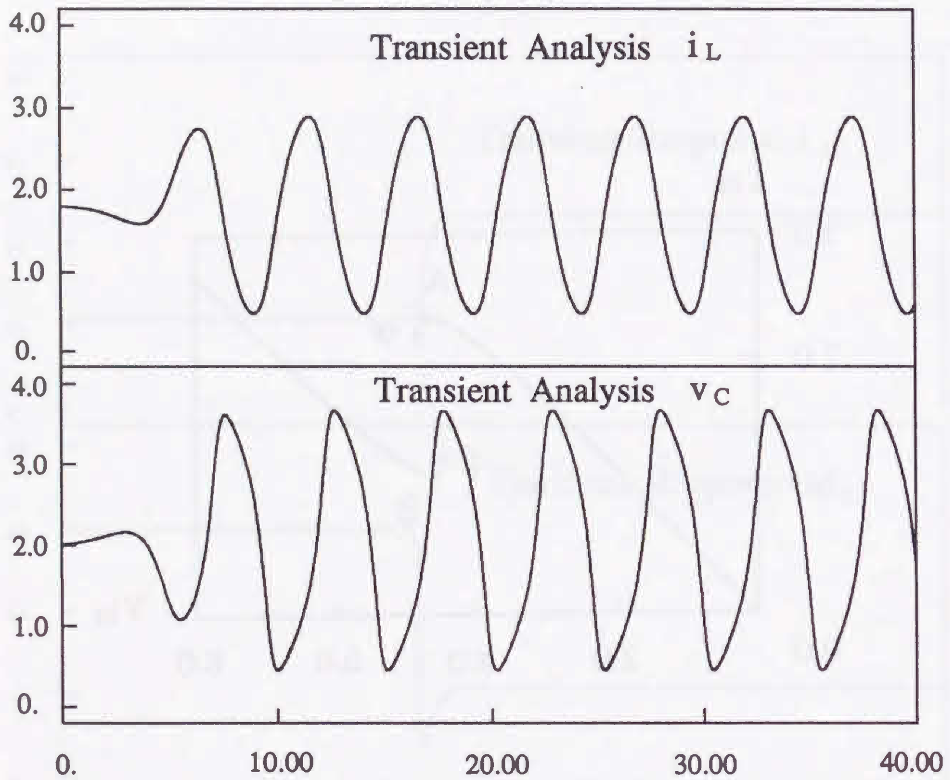
(c) Equilibrium point between (β, B)

(to be continued)



(d)Stability of DP characteristic curve for large R

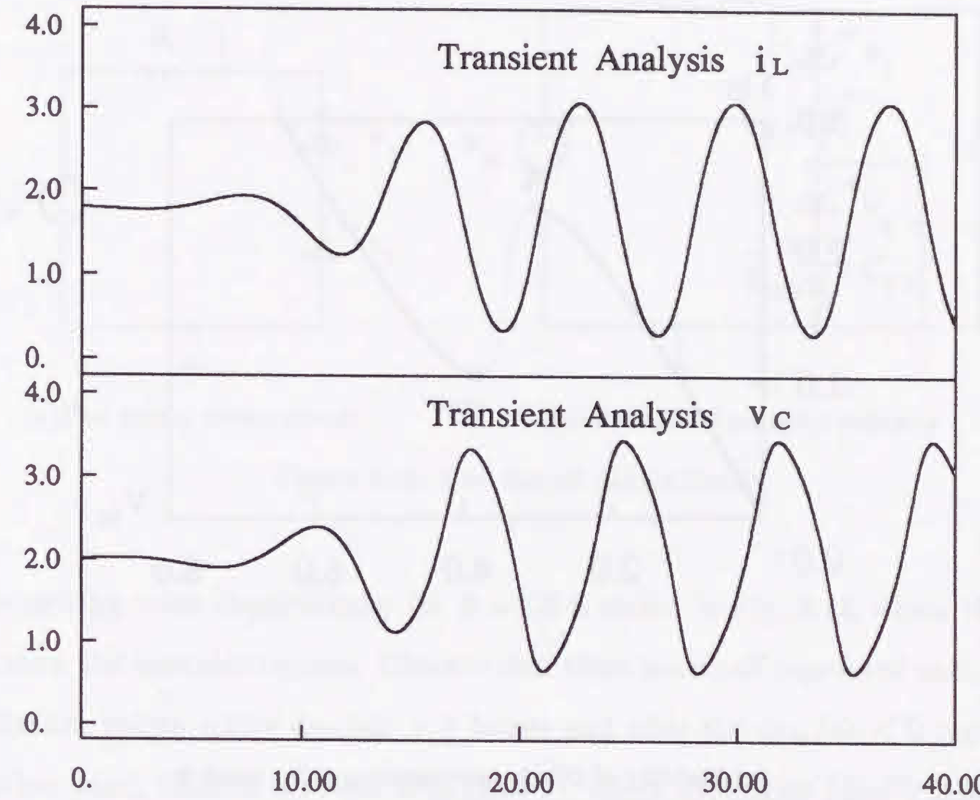
Figure 3.9: Transient analysis to one tunnel diodes circuit: type 1



vc=2., il=1.8, R=0.4, Gv=-1.04, Cp=0.5, Lp=1.

(a) Equilibrium point between (A, B)

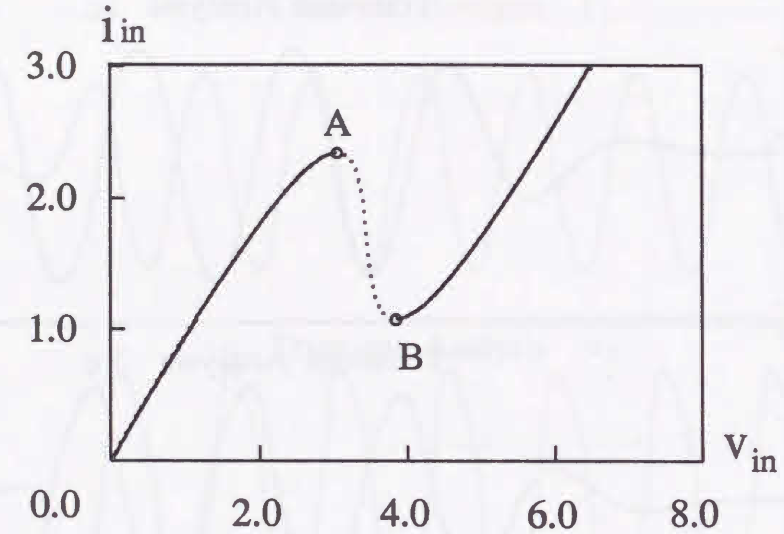
(to be continued)



vc=2., il=1.8, R=0.4, Gv=-1.04, Cp=1., Lp=1.

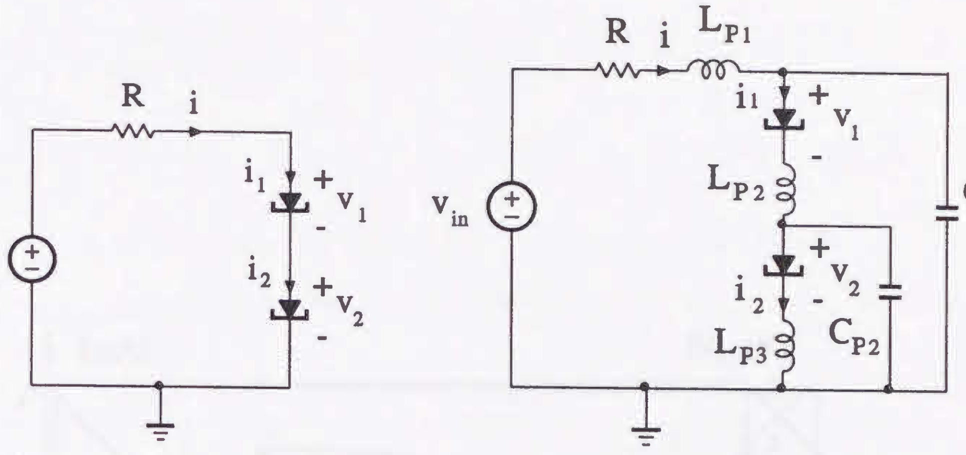
(b) Between (A, B) for different parameters

(to be continued)



(c) Stability of DP characteristic curve for small R

Figure 3.10: Transient analysis to one tunnel diodes circuit: type 2



(a)Two tunnel diodes circuit (b)Inserstion of parasitic elements

Figure 3.11: Two tunnel diodes circuit

The driving point characteristic for $R = 1.5$ is shown in Fig. 3.13, where the dotted lines show the unstable regions. Observe that there are small regions of unstable focal equilibrium points where $dv_{in}/ds > 0$ before and after the $dv_{in}/ds < 0$ regions. On the other hand, there is a closed loop $(EaFb)$, where the region (EaF) is stable and (FbE) unstable. Note that once the stability is known by *Theorem 3.1* at a point on the closed loop, the whole of the stability can be known by *Theorem 3.2.*

3.5.2 Transistor circuit

Now, consider a Flip-Flop circuit as shown in Fig. 3.14(a),

where $R_1 = R_2 = 15k\Omega$, $R_3 = R_4 = 140k\Omega$, $R_5 = R_6 = 10k\Omega$, $R_7 = 0.5k\Omega$, $C_1 = C_2 = 100pF$.

Using Ebers-Moll model for two transistors, the circuit equations can be written as

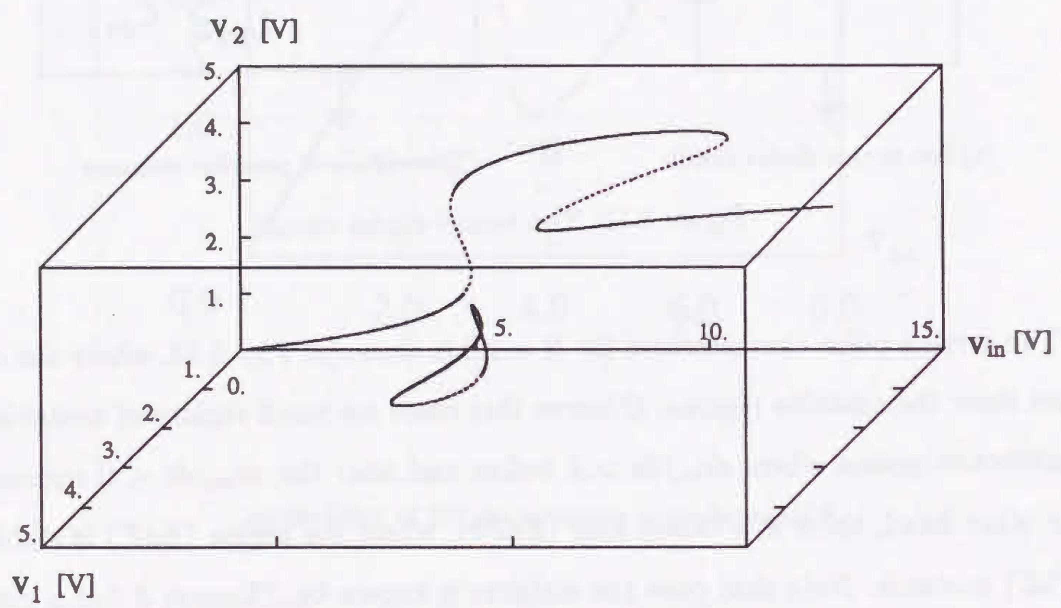


Figure 3.12: Stability of transfer characteristic for two tunnel diodes circuit

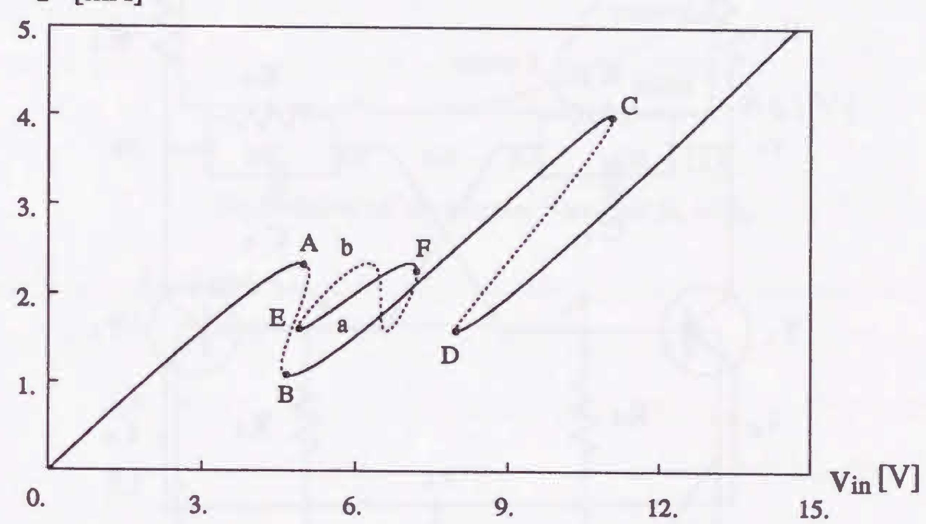
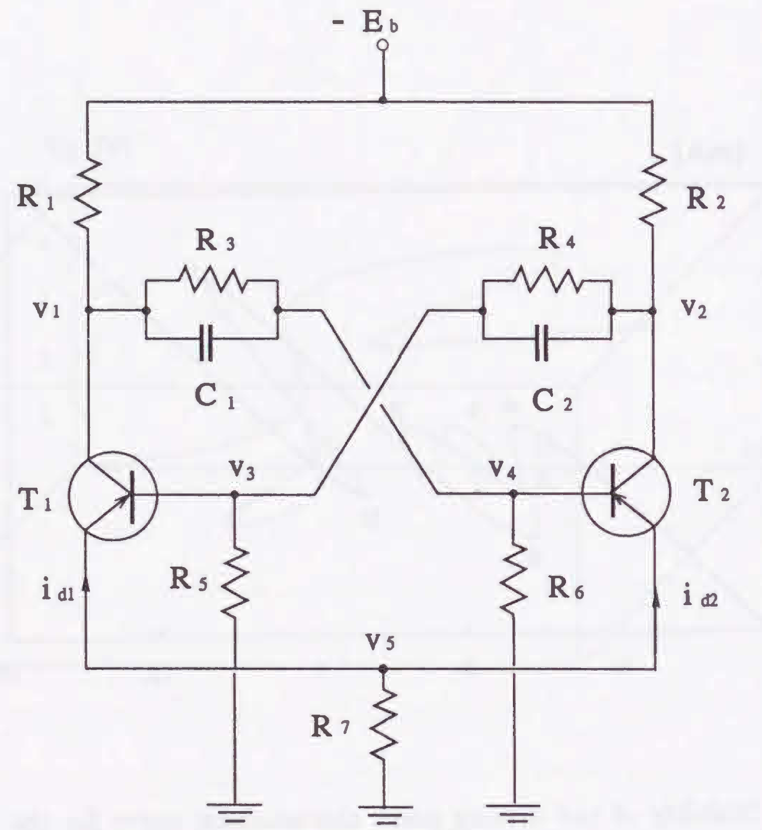
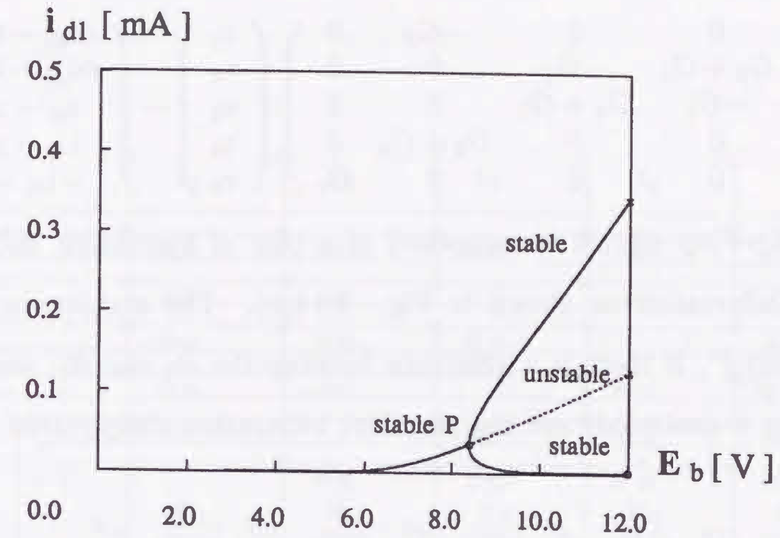


Figure 3.13: Stability of the driving point characteristic curve for the two tunnel diodes circuit

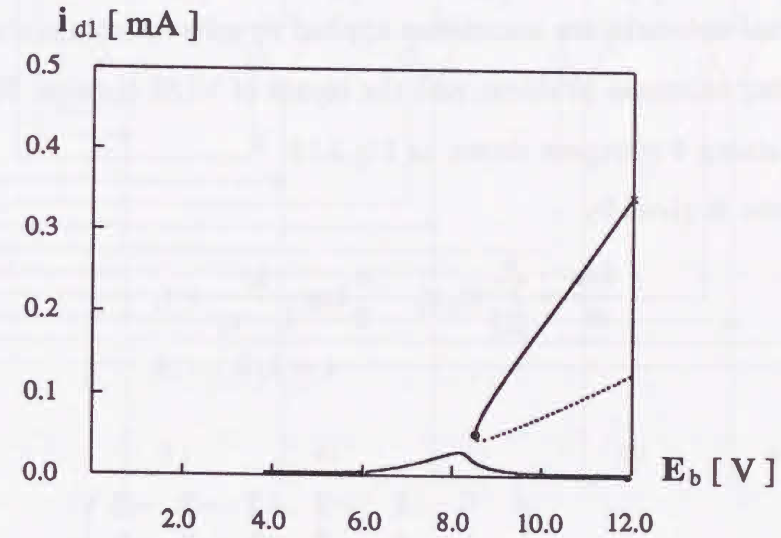


(a) Flip-Flop circuit

(to be continue)



(b) Stability of the solution curve for $R_5 = R_6$



(c) Stability of the solution curve for $R_5 = R_6 + \Delta R$

Figure 3.14: Stability of the solution curve for Flip-Flop circuit

follows:

$$\begin{pmatrix} G_1+G_3 & 0 & 0 & -G_3 & 0 \\ 0 & G_2+G_4 & -G_4 & 0 & 0 \\ 0 & -G_4 & G_4+G_5 & 0 & 0 \\ -G_3 & 0 & 0 & G_3+G_5 & 0 \\ 0 & 0 & 0 & 0 & G_7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} - \begin{pmatrix} \alpha i_{d1} - G_1 E_b \\ \alpha i_{d2} - G_2 E_b \\ i_{d1} - \alpha i_{d1} \\ i_{d2} - \alpha i_{d2} \\ -i_{d1} - i_{d2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the Flip-Flop circuit is composed of a pair of symmetric sub-circuits, it has a pitchfork bifurcation as shown in Fig. 3.14(b). The stability is changed at the pitchfork point P. If there is a difference between the R_5 and R_6, then the symmetry of the circuits is destroied, and the pitchfork bifurcation disappeared as shown in Fig. 3.14(c).

3.5.3 Hopfield neural network

Hopfield neural networks are sometimes applied to solve combinatorial problems such as the traveling salesman problem, and the layout of VLSI circuits. Now, consider the circuits containing 6 synapses shown as Fig.3.15. [2]

The equation is given by

$$\frac{du_i}{dt} = \sum_{j=1}^{6} w_{ij} x_j - \frac{a}{2} \log \frac{x_i}{1 - x_i} + I_i$$
$$i = 1, 2, \ldots, 6$$

where

$$W = \begin{pmatrix} 0 & 1 & -2 & -2 & -2 & -2 \\ 1 & 0 & -2 & -2 & -2 & -2 \\ -2 & -2 & 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & -2 & 0 & 1 \\ -2 & -2 & -2 & -2 & 1 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 3.5 & 3.5 & 5.0 & 5.0 & 3.5 & 3.5 \end{pmatrix}^T$$

[2]The example is given by Prof. A.Sakamoto at Tokushima university.

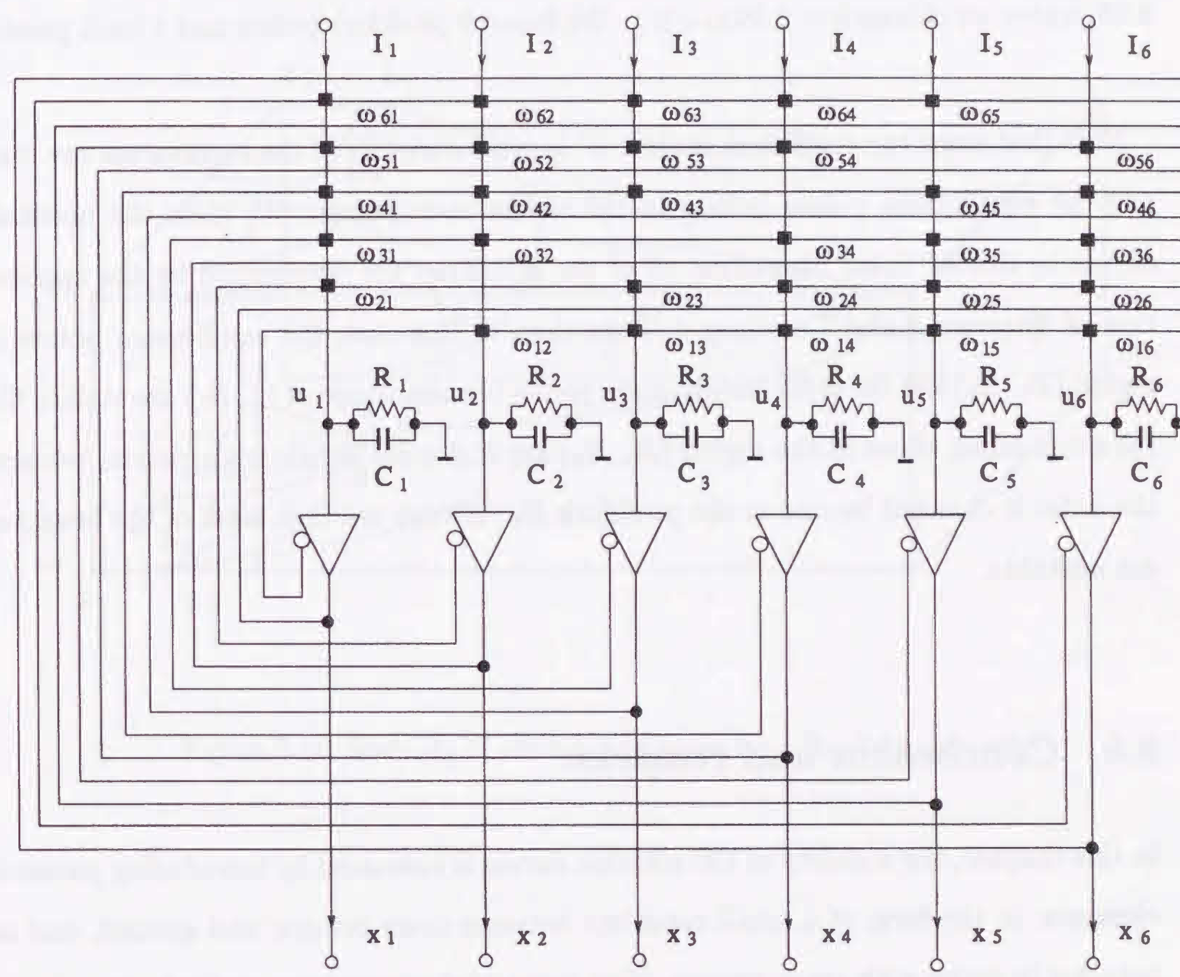


Figure 3.15: A Hopfield circuit containing 6 synapses

Setting $du_i/dt = 0$, the stationary solutions are obtained. Choosing a as an additional variable, we have a set of 6 algebraic equations with 7 variables. The solution curves are obtained starting from $a = 0.1$ [5]. The curves in the (x_1, x_3, x_7)-plane are shown in Fig. 3.16, where we choose $a = 0.29x_7 + 0.1$. We found 9 pitchfork points and 4 limit points.

Note that since the coefficient matrix W is symmetric, all of the eigenvalues are real, and the equilibrium points belong to the saddle-node points. We show the unstable curves by dotted lines. Therefore, all of the stabilities are determined by the application of *Theorem 2* and *Corollary 1*. Note that, in this case, the equilibrium points in region (B_1, B_2) are 1st order saddle-node points because those of (S, B_1) are stable. On the other hand, those in the region (B_2, B_3) are 2nd order saddle-node points, because the order is changed by one at the pitchfork B_2. We can see that most of the branches are unstable.

3.6 Conclusions and remarks

In this chapter, the stability of DC solution curves is examined by introducing parasitic elements, in the form of a small capacitor between every resistor and ground, and an inductor in series with every resistor. The ratios of the capacitors and inductors play a very important role in the stability. We have assumed that the ratios C_p/L_p may take any value from zero to infinity.

We have proved three theorems and one corollary which are very useful for checking the instability regions of the solution curves. We have two main results. The first result is that instability may be occurred in the negative differential resistance(NDR) regions

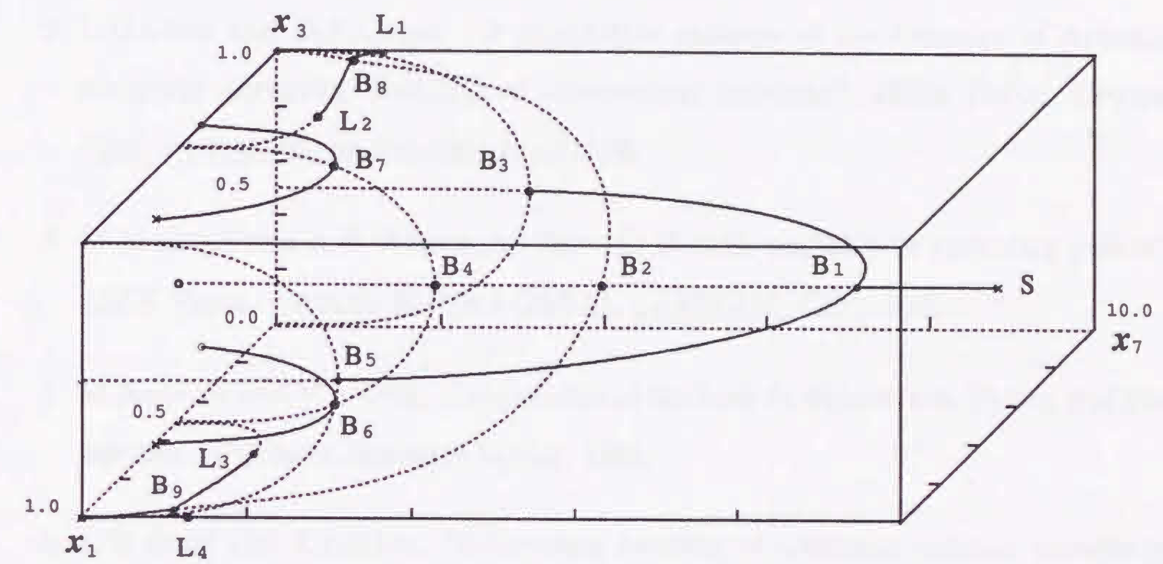


Figure 3.16: Stability of the solution curve for a Hopfield network

depending on the choice of parasitic elements. The second result is that the stability is also changed at the bifurcation points such as a turning, branch and pitchfork point. Thus, we can easily find the instability regions of the solution curves without investigating the variational equation.

References

1. L.O.Chua and G.R.Alexander, "The effects of parasitic reactances on nonlinear networks", *IEEE Trans. Circuits Syst.*,vol.CT-18, pp.520-532, May 1971.

2. L.O.Chua and D.N.Green, "A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks", *IEEE Trans. Circuits Syst.*,vol.CAS-23, pp.355-530, June 1976.

3. M.M.Green and A.N.Willson,Jr, "How to identify unstable dc operating points", *IEEE Trans. Circuits Syst.*,vol.CAS-39, pp.821-832, Oct. 1992.

4. M.Kubicek and M.Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures*, Springer-Verlag, 1983.

5. L.G.Jiang and A.Ushida, "Bifurcation analysis of nonlinear resistive circuits by curve tracing method", *IEICE Trans.*,vol.E78-A, pp.1225-1232, Sept. 1995.

6. L.O.Chua and P.M.Lin, *Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques*, Prentice-Hall Englewood Cliff, N.J., 1975.

7. R.Bellman, *Introduction to Matrix Analysis* , Tata McGraw-Hill, 1974.

8. L.A.Zadeh, and C.A.Desoer, *Linear System Theory: The State Space Approach*, McGraw-Hill, 1963.

9. R.K.Brayton, L.O.Chua, J.D.Rhodes and R.Spence, *Modern Network Theory: An Introduction*, Georgi, 1978.

10. M.Hasler and J.Neirynck, *Nonlinear Circuits* Arthech House, Inc, 1986.

11. C.S.Beightler, D.T.Phillips and D.J.Wild, *Introduction of Optimization*, Prentice-Hall, 1979.

12. D.M.Wolf and S.R.Sanders, "Multi-parameter homotopy method for finding periodic solutions of nonlinear circuits", Proc. ISCAS'94, London, pp.6.137-140, 1994.

13. L.V.Kolev, "Find all solutions of nonlinear resistive equation via interval analysis", *Int. J. of Circuit Theory Appl.*, vol.12, pp.173-178, 1984.

14. K.Yamamura and M.Ochiai, "An efficient algorithm for finding all solutions of piecewise-linear circuits", *IEEE Trans. Circuits Syst.*, vol.CAS-39, pp.213-221, March 1992.

15. A.Ushida and L.O.Chua, "Tracing solution curve of nonlinear equations with sharp turning points", *Int. J. of Circuit Theory Appl.*, vol.12, pp.1-21, 1984.

16. Y.Inoue, "Practical step length algorithms for tracing solution curves", *IEICE Trans.*, vol.J78-A, pp.381-390, March 1995.

17. Y.Shinohara, "A geometric method of numerical solution of nonlinear equation and error estimation by Urabe's proposition", *RIMS Kyoto University*, vol.5, pp.1-9,1969.

18. A.H.Nayfeh and D.T.Mook, *Nonlinear Oscillations*, John Wiley, New York, 1979.

19. L.O.Chua and A.Ushida, "A switching-parameter algorithm for finding multiple solutions of nonlinear resistive circuits", *Int. J. of Circuit Theory Appl.*, vol.4, pp. 215-239, 1976.

20. A.I.Mees, *Dynamics of Feedback Systems* John Wiley and Sons, 1981.

21. M.M.Green and A.N.Willson,Jr "On the relationship between negative differential resistance and stability for nonlinear one-ports", *Proc. IEEE Int. Symp. on Circuits and Systems*, pp.2474-2477, May, 1993.

22. M.M.Green and A.N.Willson,Jr "On the relationship between negative differential resistance and stability for nonlinear one-ports", *IEEE Trans. on Circuits and Systems - Part I*, vol.43, pp.407-410, May, 1996.

Chapter 4

Overall Conclusions

In this study, we researched the methods and algorithms for effectively tracing the characteristic curves of nonlinear resistive circuits and distinguishing their stabilities.

It is well known that the characteristic curves are solution curves for a set of DC circuit equations which have n equations in $(n+1)$ variables. On characteristic curves there are many kinds of bifurcation points such as turning point, branching point and isolated point. knowning the location of bifurcation points is very important to understand the behaviors of circuit and effectively trace the characteristic curves. Since the rank of Jacobian matrix of circuit equations is decreased by one or more at these points, we cannot apply usual technique such as Newton-Raphson method to calculate them.

In this paper we have shown that if we modify the original equations such that the singularity is removed for the augment system, which consist of the original equations and some additional equations, then these points can be still found by Newton-Raphson method. This numerical method is termed as direct method. We also shown here that our curve tracing algorithm can continuously trace the solution curves having the turning points and/or branching points, and proposed some simple sign testing to

determine the solution curve whether past through the bifurcation point in this case, so that the locations of bifurcation points can be decided by using the curve tracing algorithm , which is called the indirect method. To obtain the whole solution curves, we discussed the calculating methods for determining the directions of branches at the branching points, which are very important in order to continuously tracing the solution curves. Combining these algorithms, complicated solution curves can be easily traced by the curve tracing method.

A lot of numerical computation examples have been given to exhibit the efficiency of our curve tracing algorithm. It is shown that the algorithm can be used successfully to tracing the solution curves of diode circuit, transistor circuit and Hopfield circuit which are all strong nonlinear circuits, at same time, the bifurcation points on the solution curves can also be located.

Distinguishing the stability of characteristic curves for nonlinear resistive circuits is requirement and importance to design various electronic circuit exactly.

Since every resistive element has a parasitic component, solutions on the characteristic curves are stable or unstable. We have examined the stability of DC solution curves for nonlinear resistive circuit by introducing parasitic elements, in the form of a small capacitor between every resistor and ground, and an inductor in series with every resistor. It is found that the ratios of the capacitors and inductors play a very important role in the stability.

We have proposed four definitions, proposed and proved three theorems and one corollary based on Liapunov's direct method and our curve tracing method, which are very useful to check the instability regions of the solution curves. Since the stability is *mainly* changed at the boundary of the presence of negative differential resistance

(NDR) and the bifurcation points such as turning and pitchfork points on the DC characteristic curves, so that the instability regions of the solution curve are easily found by determining both the locations of bifurcation points and NDR regions of the nonlinear resistors.

We gave some numerical examples to show these methods and theorems. The transient analysis based on Runge-kutta method has been done to verify the exactitude of these theorems.

All methods and algorithms proposed in the paper would help one to analysis and design electronic circuits effectively.

Acknowledgements

This thesis is a collection of studies carried out under the direction and guidance of Professor Akio Ushida of Tokushima University during 1993-1996. Therefore, I would like to take this opportunity to thank all those who helped me and supported me to accomplish my work.

Five years ago, I came to Tokushima University as a foreign student. After two years, I finished the master course in electrical engineering under the direction and guidance of Professor Akio Ushida. Since October of 1993, I have been working for Ph.D. degree in electronic system engineering. I would like to express my great gratitude to Professor Akio Ushida for supplying me with this research and study chance, and also for his outstanding guidance, helpful advice and continuous encouragement during this period. I could not imagine that this study could be successful without his guidance.

I wish to greatly acknowledge my indebtedness to Professor Kiyoshi Kohno, the previous director of Faculty of Engineering, and his wife Mrs. Nayoko Kohno who was my first Japanese language lecturer after I came to Japan. As very nice teachers, seniors and friends, they give me and my husband much help and very warm encouragement. I will never forget their important support and great help.

I sincerely thank Professor Akio Sakamoto, Associate Professor Takashi Shimamoto and Dr. Yoshifumi Nishio for their helpful advice and kind encouragement throughout the work.

I wish to express my appreciation to Professor Hiroshi Kawakami and Professor Yosuke Kinouchi of Tokushima University, the members of my thesis committee, for their comments and suggestions, and particularly for their careful and critical reading of this dissertation.

I wish to express grateful acknowledgement to Professor Tetsuo Nishi at Kyusyu University, Professor Norio Yamamoto at Kyusyu Institute of Technology and Associate Professor Kiyoshi Yamamura at Gunma Univer sity for their useful suggestions and comments.

I am very much grateful to Mr. Yoshihiro Yamagami of Tokushima University for his very useful help in using computers and warm encouragement.

I have enjoyed an interesting research life with my colleagues of Ushida Laboratory and Sakamoto Laboratory and I am deeply indebted to them. I am very grateful to Mr. Shinzao Liu, Mr. Yuichi Tanji, Mr. Yoshinobu Setou, Mr. Takashi Sugimoto, Mr. Masahiro Wada, Mr. kenichi Suga, Mr. Jyun Kishida, Mr. Tatsuki Okamoto, Mr. Takumi Ichihara, Mr. Toshihisa Oohiro, Mr. Brett Chandler and other people for their friendly help during the period of this research. I also wish to thank all members in Ushida and Sakamoto Laboratory, Department of Electrical and Electronic Engineering, Tokushima University, for a warm and interesting research environment which I have enjoyed in Ushida and Sakamoto Laboratory.

I wish to express grateful acknowledgement to Professor Katsuo Isaka who was the foreign student director of Faculty of Engineering and all other related people of Tokushima University, for their kind help and warm support.

I would like to express my deep appreciation to my husband Chen He for his affectionate continuous encouragement and great help.

At last, but not least, I would like to express my gratitude to my father, my mother, my husband's father, my husband's mother and my son who helped and encouraged me greatly, and gave all kinds of assistance to obtain my educational aims under the most comfortable atmosphere and to make this study a great success.

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Appendix A

A List of the Related Papers by the Author

A.1 Publications

1. Lingge JIANG and Akio Ushida, "Bifurcation Analysis of Nonlinear Resistive Circuits by Curve Tracing Method" *Institute of Electronics, Information and Communication Engineers Trans.*, vol.E-78A, no.9, pp.1225-1232, Sept. 1995.

2. Yuichi TANJI, Lingge JIANG and Akio Ushida, "Analysis of Pulse Responses of Multi-Conductor Transmission lines by a Partitioning Technique" *Institute of Electronics, Information and Communication Engineers Trans.*, vol.E-77, no.12, pp.2017-2027, Dec. 1994.

A.2 International Conferences

1. Lingge JIANG and Akio Ushida, "Stability of Characteristic Curves of nonlinear resistive circuits", *Proc. 1995 Int. Symp. on Nonlinear Theor. and its Appl.*, vol.2, no.2, pp.1165-1170, Las Vegas, America, Dec. 1995.

2. Akio Ushida and Lingge JIANG "Analysis of Multi-Conductor Transmission lines Terminated by Nonlinear Subnetworks " *Proc. 1993 Joint Technical Conference on Circuits/Systems, Computers and Communications*, vol.1, no.2, pp.537-542, Nara, Japan, July, 1993.

A.3 Technical Reports and Other Presentations

1. Lingge JIANG and Akio Ushida, "Stability of Nonlinear Resistive Circuits having Parasitic Elements" *Technical Report of the Institute of Electronics, Information and Communication Engineers*, vol.NLP95-99, pp.7-14, Mar. 1996.

2. Akio Ushida and Lingge JIANG, "Transient Analysis of Nonlinear Distributed Circuits by Relaxation Hybrid Harmonic Balance Method" *Technical Report of the Institute of Electronics, Information and Communication Engineers*, vol.NLP92-11, pp.35-40, July, 1992.

3. Lingge JIANG and Akio Ushida, "Stability of Nonlinear Reciprocal Resistive Circuits with Parasitic Elements" *Proc. of IEICE 1996 National Conference*, vol.1, p.98, Tokyo, Mar. 1996.

4. Lingge JIANG and Akio Ushida, "Analysis of Bifurcation Phenomena of Neural Network" *Proc. of IEICE 1995 National Conference* vol.1, p.92, Fukuoka, Mar. 1995.

5. Lingge JIANG and Akio Ushida, "Transient Analysis of Nonuniform interconnects by the Chebyshev Expansion Method" *Proc. of IEICE 1994 National Conference*, vol.1, p.55, Yokohama, Mar. 1994.

様式 9

論文審査の結果の要旨

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学位論文題目

Characteristic Curves of Nonlinear Resistive Circuits and their Stabilities

審査結果の要旨

電子回路の直流解析は回路設計における最も重要な問題の一つである。この場合、回路方程式は非線形連立代数方程式で記述され、この方程式の全ての解を求めることが必要となる。このような問題は古くて新しいテーマであり、工学や数学の分野の多くの研究者によって盛んに研究が行われている。

一般に、$n+1$ 個の変数をもち n 個の方程式から構成されている非線形連立代数方程式の解は $n+1$ 次元空間における解曲線となる。ここで、x_{n+1} として、強制入力を選ぶならば、これを満足する解は駆動点特性曲線とか伝達特性曲線に対応する。また、x_{n+1} を補助変数とみなして、非線形連立代数方程式を解析する手法はニュートン法に比べて収束領域が広いために良く用いられる手法の一つである。

このような解曲線上には各種の分岐点が存在する。特に、交差点では解曲線が複数個に分岐するが、このような解曲線を追跡するためには分岐点において枝の方向を決定し、再び解曲線を追跡する必要がある。

一方、全ての抵抗素子には寄生素子が存在する。寄生素子値は抵抗素子値に比べて十分小さいと仮定できるが、大きさよりも、その存在が解曲線の安定性に関して非常に重要な役割を果たす。本研究では、さらに、分岐点において解曲線の安定性が変わることに注目し、解曲線の安定性に関する問題について議論している。

第二章では、弧長法を用いた解曲線追跡法を提案している。この場合、非線形連立代数方程式は微分代数方程式に変換され、これをステイフな系に強い後退差分公式を導入して解曲線を追跡しようと云うのが本求解法である。この場合、変曲点のような分岐点を含んだ曲線は連続的に追跡できるが、交差点、孤立点等が存在する場合が問題である。このような場合に、まず、正確な分岐点の位置を計算し、方向ベクトルを求めて再出発をする方法を提案している。このようにして全ての解曲線を連続的に追跡できる。

第三章は、解曲線の安定性について議論したものである。抵抗回路には寄生キャパシタやインダクタが必ず存在する。それらは解の安定性に関して非常に重要な役割を果たす。従来から平衡点の安定性に関しては、変分方程式の固有値を計算することにより判定する方法や、係数行列の行列式の値から安定性を判定する方法を提案が提案されている。これらは、特性曲線の各点で安定判別を行う必要があるためあまり有効な手法とは云えない。本論文では、二つの安定判別手法を提案している。その一つは回路網トポジーとリアプノフの安定性に基づいたものであり、もう一つは二章の解曲線追跡手法に基づいている。これらの安定判別法は解曲線追跡法に直接関係しているので、特性曲線の形状から安定性を容易に判定できるという利点がある。

以上、本研究は電子回路の直流解曲線の求解法と、得られた解に対する安定判別法を議論したものであり、回路設計に非常に有効である。したがって、本論文は博士 (工学) の学位授与に値するものと判定する。